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Technical Report

Optimization Of The Min-Plus Convolution Computation Under Network Calculus Constraints

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Abstract

Network Calculus is a system theory for deterministic queueing systems. The min–plus convolution is an operation that is used for computations in network calculus. But until now there are few results on computing this convolution efficiently. Being able to do this is of great importance in order to make the application of network calculus more widespread. Therefore, this issue is targeted in this report. We give a brief overview over the basics of network calculus, introducing some basic operations in the min–plus algebra. Then transforms used to compute the classical convolution in the usual algebra are reviewed and basic results are listed. In the following we attempt to derive similar results for the min–plus convolution. We make use of the Fenchel transform, a tool used in convex analysis.

Finally, we make use of the property, that in network calculus one usually deals with piecewise linear functions. For convex functions good results exist. Since there are very few results in the nonconvex case, we first analyse these by a brute force method, before pointing out the general equation for a certain type of functions, which are often encountered in Network Calculus.

1 Introduction

Improving Quality of Service (QoS) in the Internet has become a significant issue in recent years. The most common QoS architectures are Integrated Services [3] and Differentiated Services [1].
For modelling and evaluating QoS enabled networks, there is a concept that begins to gain more and more importance in judging and improving network performance. This tool is referred to as *network calculus*. It is a more theoretical (and systematical) approach that tries to apply methods known from classical system theory to computer networks. The first concepts of network calculus were developed in a series of papers by Cruz, essentially in [5] and [6].
In [2] the study of network caculus is motivated as follows:
"Network Calculus is a set of recent developments that provide deep insights into flow problems encountered in networking. The foundation of network calculus lies in the mathematical theory of dioids, and in particular the Min-Plus dioid, which is also known as Min-Plus algebra. With network calculus, we are able to understand some fundamental properties of integrated services networks, window flow control, scheduling and buffer or delay dimensioning."
Network calculus is a theory of deterministic queueing systems found in computer networks. It can also be viewed as the system theory that applies to computer networks. The main difference with traditional system theory, as the one that was so successfully applied to design electronic circuits, is that here we consider another algebra, namely the Min-Plus algebra.
As in conventional system theory, one can define a convolution on this algebra, which turns out to be very helpful in theory. Investigating this so called min–plus convolution is subject of this report.
This report is outlined as follows. We first give an introduction to Network Calculus, Min-plus Algebra, and some transforms from conventional system theory. This is followed by a review of some related work and pointing out of some existing result. We then describe our ideas and approaches. Finally, we analyse a special case often encountered in Network Calculus and develop a general equation to calculate its min–plus convolution.

1.1 Network Calculus Basics

The following definitions are taken out of [2].

Input and output functions

Definition: $R(t)$ is called an *input function*, if $R(0) = 0$ and R is wide–sense increasing, that is for all $t_1 \leq t_2$ holds $R(t_1) \leq R(t_2)$.

$R(t)$ denotes the number of bits arriving in the interval $[0, t]$. The time t and $R(t)$ can either be discrete or continuous. If both t and $R(t)$ are continuous, we say we have a *fluid model*. There are mappings between continuous and discrete time models.

Definition: $R^\circ(t)$ is called an *output function* for a system S, if it cumulates the output (in bits) of S in the interval $[0, t\,]$ for all $t \geq 0$.

In this definition S might be a single buffer, a network node or a complete network. As above, t and $R^\circ(t)$ may be discrete or continuous.
For a system S it should (obviously) hold $R^\circ(t) \leq R(t)$; this assumptions implies in particular that we have a lossless system without routing loops.

Arrival curves

We want to formulate necessary conditions for service guarantees. One approach is to place constraints on the arrival process of flows. Enforcing this constraint is called *traffic regulation*. In order to abstract from a specific traffic regulation algorithm we introduce the concept of arrival curves.

Definition: A wide–sense increasing function α is called an *arrival curve* for an input function $R(t)$, if $\alpha(t) = 0$ for $t < 0$ and for $t \geq 0$

$$R(t) - R(t - s) \leq \alpha(s) \qquad \forall\, 0 \leq s \leq t$$

holds. We also say R is *α–smooth* or R is constrained by α.

Service curves

Arrival curves constrain the flow arrival process. To offer guarantees on flow characteristics such as maximum delay, throughput or zero loss network nodes need to allocate capacity to flows. An important mechanism here is the scheduling strategy. In order to abstract from a specific packet scheduling discipline we introduce the concept of service curves.

Definition: Consider a system S and a flow through S with input function R and output function R°. We say S offers a *service curve* β to the flow, if β is wide–sense increasing and

$$R^\circ \geq R \otimes \beta$$

Here the operation $\otimes$ denotes the min–plus convolution in the Min–Plus algebra.
An important observation is theorem 1.4.6 in [2]:

Theorem 1.4.6 Assume a flow traverses systems S_1 and S_2 in sequence. Assume S_i offers a service curve β_i, $i = 1, 2$ to the flow. Then the concatenation of the two systems offers a service curve of $\beta_1 \otimes \beta_2$ to the flow.

1.2 Min–plus Algebra

The standard algebra is defined on the triple $(\mathbb{R}, +, \cdot)$. In contrast, the min–plus algebra is defined on the triple $(\mathbb{R} \cup \infty, \min, +)$, or $(\mathbb{R} \cup \infty, \oplus, \odot)$ with $\oplus = \min$ and $\odot = \cdot$. One can show, that the min–plus algebra is a *dioid* or a *semi–ring*. That is we have

- closure of $\oplus$:

$$\forall\, a, b \in \mathbb{R} \cup \infty \qquad \Longrightarrow \qquad a \oplus b = \min\{a, b\} \in \mathbb{R} \cup \infty$$

- associativity of $\oplus$:

$$(a \oplus b) \oplus c = \min\{\min\{a, b\}, c\} = \min\{a, \min\{b, c\}\} = a \oplus (b \oplus c)$$

- existence of a zero element for $\oplus$:

$$\forall a \in \mathbb{R} \cup \infty \qquad \Longrightarrow \qquad a \oplus \infty = \min\{a, \infty\} = a$$

- idempotency of min:

$$\forall a \in \mathbb{R} \cup \infty \qquad \Longrightarrow \qquad \min\{a, a\} = \min\{a\}$$

- commutativity of $\oplus$:

$$\forall\, a, b \in \mathbb{R} \cup \infty \qquad \Longrightarrow \qquad a \oplus b = \min\{a, b\} = \min\{b, a\} = b \oplus a$$

- associativity of $\odot$
- absorption of zero element for $\odot$
- neutral elemet for $\odot$
- distributivity of $\odot$ with respect to $\oplus$

These characteristics define a commutative dioid.

The min–plus algebra is different from the algebra we are used to. We will illustrate this with a few examples:

1. We are used to having a unique inverse operation with respect to the common addition:

$$a \oplus b = c$$

With the interpretation of $\oplus = +$, we get $b = c - a$, which is obviously the unique solution. In the min–plus algebra we interpret $\oplus$ as min, so the above equation reads

$$\min\{a, b\} = c$$

There are various cases for the choice of b, assuming a and c are given. For simplicity, we show the effects chosing different numbers for a and c.

   a) $a = 1$ and $c = 0$. In this case b is uniquely determined, if equality should hold: $b = 0 = c$.

   b) $a = -1$ and $c = 0$. In this case a solution to the equation does *not* exist, no matter if we chose $b \geq a$ or $b < a$!

   c) $a = 0$ and $c = 0$. In this case we can take $b \geq 0 = a$, yielding infinitely many solutions to the equation. This can be fixed by taking the smallest b that satisfies the equation. In this case we would take $b = 0$. b is referred to as smallest approximation from above.

The latter argument can be extended to much more general equations and even systems of equations, where one can define a solution operator as the greatest lower bound or the least upper bound; we won't need these operators in this paper, so for details please refer to e.g. [7].

2. In the conventional algebra we have

$$a \oplus a = 2a$$

that is $a + a = 2a$. In the min–plus algebra we have by the idempotency of the addition:

$$a \oplus a = a$$

Furthermore consider the formula

$$a \oplus (2 \odot a)$$

In the conventional sense, that is with $\oplus = +$ and $\odot = \cdot$, we have

$$a + 2 \cdot a = 3a$$

If we take $\oplus = \min$ and $\odot = +$, we have

$$a \oplus (2 \odot a) = \min\{a, a + 2\} = a$$

which is in general not equal to $3a$.

3. In the min–plus algebra expressions of the form

$$b \odot x^m, \qquad m \in \mathbb{N}$$

(with $x^m = \underbrace{x \odot \ldots \odot x}_{m-times}$) can be understood as piecewise linear functions in the conventional sense, since $\odot = +$:

$$b \odot x^m = m \cdot x + b$$

This is a helpful interpreation, if we want to investigate a power–series in one variable in the min–plus algebra. Geometrically (in two space dimensions) we can easily evaluate expressions like

$$f(x) = \bigoplus_{i \in I} a_i \odot x^i$$

if I is a finite set, $a_i \in \mathbb{R}$. We simply draw the piecewise affine functions $i \cdot x + a_i$ and determine the minimum at each point $x \in \mathbb{R}$.

Remark 1.1 Convexity will prove to be very helpful for us; its formal definition is postponed to the next chapter. We just note that it is immediately clear that f can not be convex.

We hope that the examples just given suffice to show that one has to get used to the min–plus algebra and that easy computations in the conventional algebra do not carry over immediately to the min–plus algebra!

Min–plus convolution

We want to define a convolution for special functions $f : \mathbb{R} \to \mathbb{R}$. Let

$$\mathcal{F} := \{f : \mathbb{R} \to \mathbb{R} \mid \text{supp}(f) \subseteq \mathbb{R}_0^+, f(t_1) \le f(t_2) \quad \forall t_1 \le t_2\}$$

denote the set of nonnegative wide–sense increasing functions. Then define for $f, g \in \mathcal{F}$ and $t \ge 0$ the min–plus convolution by

$$\otimes : \mathcal{F} \times \mathcal{F} \to \mathcal{F} \qquad (f \otimes g)(t) := \inf_{0 \le s \le t} \{f(t-s) + g(s)\}$$

The min–plus convolution has the following properties, as is shown in [2]:
Let $f, g, h \in \mathcal{F}$.

- closure
$$(f \otimes g) \in \mathcal{F}$$
- associativity
$$(f \otimes g) \otimes h = f \otimes (g \otimes h)$$

- zero element for min is absorbing for $\otimes$

$$f \otimes \epsilon = \epsilon$$

  with $\epsilon(t) = +\infty$ for $t \geq 0$ and $\epsilon(t) = 0$ otherwise.

- existence of a neutral element

$$f \otimes \delta_0 = f$$

  with $\delta_0(t) = \infty$ for $t > 0$, $\delta_0(t) = 0$ otherwise.

- commutativity

$$f \otimes g = g \otimes f$$

- distributivity with respect to min

$$(f \oplus g) \otimes h = (f \otimes h) \oplus (f \otimes h)$$

- invariance to addition of a constant
  For any $K \in \mathbb{R}^+$ we have

$$(f + K) \otimes g = (f \otimes g) + K$$

Further one can show that $(\mathcal{F}, \min, \otimes)$ is a dioid and $\otimes$ is a linear operation on $(\mathbb{R}, \min, +) = (\mathbb{R}, \oplus, \odot)$.

1.3 Laplace and Fourier Transform

A basic operation in system theory is the convolution of functions f and g, which for $t \in \mathbb{R}$ is defined as

$$(f \otimes g)(t) := \int_{\mathbb{R}} f(t-s) \cdot g(s)\, ds$$

In the min-plus algebra the conventional addition is substituted by the min-operator and the conventional multiplication by the conventional addition. Being vauge mathematically, one can few the integral as an infinite sum:

$$\int_{\mathbb{R}} f(t-s) \cdot g(s)\, ds \approx \sum_{s \in \mathbb{R}} f(t-s) \cdot g(s) = \bigoplus_{s \in \mathbb{R}} (f(t-s) \odot g(s))$$

Interpreting the operations $\oplus$ and $\odot$ in the Min-Plus Algebra yields

$$(f * g)(t) := \min_{s \in \mathbb{R}} \{f(t-s) + g(s)\} \tag{1.0}$$

In (1.0) we realize the min–plus convolution.
One important operation in classical system theory is the fourier transformation, which makes the computation of the conventional convolution more pleasant.
Being more precise, we take into consideration the following results taken from [10]:

The *Laplace Transformation* for a function f is given by

$$\mathcal{L}(f)(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt$$

for those $s \in \mathbb{C}$, for which the integral converges. The set

$$R(f) := \{s \in \mathbb{C} \mid \|\mathcal{L}(f)(s)\| < \infty\}$$

is referred to as *region of convergence*.
The Laplace transform is a linear operation, that is for $s \in R(f) \cap R(g)$ and $a, b \in \mathbb{C}$ we have

$$\mathcal{L}(a \cdot f + b \cdot g)(s) = a \cdot \mathcal{L}(f)(s) + b \cdot \mathcal{L}(g)(s)$$

The *conventional* convolution of two functions f, g is given by

$$(f \otimes g)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau)\,d\tau$$

Now the Laplace transformation yields

$$\mathcal{L}(f \otimes g)(s) = \mathcal{L}(f)(s) \cdot \mathcal{L}(g)(s)$$

The *Fourier Transformation* of a function f is defined as

$$\mathscr{F}(f)(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

This is a special case of the Laplace Transformation; one simply has to take $s = j \cdot \omega$.
We have for the Fourier Transform

$$\mathscr{F}(f \otimes g)(\omega) = \mathscr{F}(f)(\omega) \cdot \mathscr{F}(g)(\omega)$$

The Fourier transform turns the complicated computation of the convolution into a multiplication of the Fourier transforms of f and g. Hence we reduced the complexity of computing the convolution, provided that we can compute $\mathscr{F}(f)$ and $\mathscr{F}(g)$ rather easily and that there is an inverse $\mathscr{F}^{-1}$.
We are trying to find a transformation for the min–plus convolution, that posesses similar properties.

2 Related Work

2.1 A Brief Literature Review

This paper was motivated mainly by the study of [2]. The min–plus convolution is introduced there as an important tool in network calculus. In a few examples its computation is illustrated in the above book.
The min–plus convolution is a natural tool in network calculus and is employed in many ways. It is used in [2] e.g. to derive bounds on the throughput and delay for various kinds of traffic (e.g. VBR and CBR); it also has its applications in the discussion and description of traffic shapers.
[4] analyzes network calculus from a different point of view. The main consideration there involves network calculus using filters.
In [7] one can find (to our knowledge) the most general treatment on the min–plus algebra. Equations and systems of equations in the min–plus algebra are treated there, as well as power series in one and two variables in a special algebra, based on the min–plus algebra. We tried to follow some ideas by introducing and briefly discussing the construction of a transformation κ_g, which (for appropriate choice of g) becomes a power series in one respectively two variables. This construction can be found in section 3.10.
In [9] one can find a discussion on modeling discrete event time systems, e.g. how equations are derived from Petri-nets and how their solution can be computed. In particular, the $\Gamma-$ Transform of a series in the max–plus algebra is defined. We tried to transfer some ideas, but it did not work for our purposes.
As opposed to [9] this paper does not discuss system theory or the solution of equations in the min–plus algebra. It is solely devoted to the study of the (efficient) computation of the min–plus convolution and the properties the convolution posesses.

2.2 Some Existing Results

In [2], chapter 3 one can find the following.

Concave, Convex and star–shaped functions

Let $0 \leq \lambda \leq 1$.

1. A function $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is *convex* if and only if for all $x, y \in \mathcal{D}$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{2.0}$$

2. A function $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is *concave* if and only if $-f$ is convex.

3. A function $f \in \mathcal{F}$ is *star–shaped* if and only if for all $t > 0$ the function $f(t)/t$ is wide–sense increasing.

For concave and convex functions the min–plus convolution has the following properties, confirm [2], Theorem 3.1.6:
Let $f, g \in \mathcal{F}$.

- If $f(0) = g(0) = 0$ then $f \otimes g \leq \inf\{f, g\}$. Moreover, if f and g are star–shaped, then $f \otimes g = \inf\{f, g\}$.

- If f and g are convex then $f \otimes g$ is convex. In particular if f and g are convex and piecewise linear, $f \otimes g$ is obtained by putting end–to–end different linear pieces of f and g, sorted by increasing slopes.

Remark 2.1 The above result seems promising, since in some cases one can find an easy formula for computing the min–plus convolution. In network calculus one uses certain classes of functions, in particular piecewise linear ones. It turns out that the above results do not need to apply to piecewise linear functions.

1. All linear functions are convex! This is obvious, since for a linear function f we have
$$f = m \cdot x + b$$
for some $m, b \in \mathbb{R}$ and one easily checks the definition:
$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= m \cdot (\lambda x + (1-\lambda)y) + b \\ &= \lambda(m \cdot x + b) + (1-\lambda)(m \cdot y + b) \end{aligned}$$
But linear functions are not star-shaped in general: Take for example $m, b > 0$, then
$$\frac{f(t)}{t} = m + \frac{b}{t}$$
This is not wide–sense increasing in t.

2. Piecewise linear functions do not have to be convex or concave. Consider for example for $U > T \geq 0$
$$f(x) = \begin{cases} 0 & x \leq T \\ m \cdot (x - T) & T < x \leq U \\ n \cdot (x - U) + m(U - T) & U < x \end{cases}$$
If $n \geq m$, the function is convex. This can either be checked by definition (2.0) or by another (equivalent) characterization of convexity introduced in the next

chapter. For $n < m$ the function is neither convex nor concave.
Further we can investigate the quotient

$$\frac{f(x)}{x} = \begin{cases} 0 & x < 0 \\ m - \frac{mT}{x} & T < x \leq U \\ n - \frac{nU - m(U-T)}{x} & U < x \end{cases}$$

For example, if $n \geq 0$ and $nU - m(U-T) < 0$ this is not wide–sense increasing in x and thus not star–shaped.

3 Our Ideas and Approaches

The first ideas were devolped during the study of [7]. In particular we were interested in the max–plus algebra and its diod–structure. We tried to map the ideas presented in chapter 5 on " Two–Dimensional Domain Description of Event Graphs " to our situation, but that did not work well. In chapter 6 we then found out that inf-convolutions are converted into pointwise conventional additions by the Fenchel–Transform, a tool used in convex analysis. This led us to the study of [8]. Furthermore this approach seemed promising, since for linear functions, which are a subclass of convex functions, one already found methods for computing the min–plus convolution more efficiently.

3.1 Basics in Convex Analysis

Unless otherwise stated, the definitions, claims, propositions and proofs appearing in this section are all taken out of chapter one in [8].

Definition 3.1 Let I be a nonempty interval of $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be *convex on I* when

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

for all pairs of points (x, y) in I and all $\alpha \in (0, 1)$.
It is said to be *strictly convex* when strict inequality holds, if $x \neq y$.

Definition 3.2 The *epigraph* of a function $f : D \to \mathbb{R}$ is

$$\operatorname{epi} f := \{(x, r) \mid x \in D, r \geq f(x)\}$$

So the epigraph of f is "everything that lies above the graph of f".
Now we can define convexity of a function f again in terms of the epigraph.

Definition 3.3 Let I be a nonempty interval of $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is convex on I if and only if $\operatorname{epi} f$ is a convex subset of $\mathbb{R}^2$.

Remark 3.4 Due to this definition we can easily check the convexity of a piecewise linear function; we simply have to decide if $\operatorname{epi} f$ is convex. Therefore we recall:
A convex subset $C \subset \mathbb{R}^2$ is a set, such that, if the points x and y are in C, then the segment joining x and y is also in C.

Claim 3.5 Let f and g be convex and set for all $x \in \mathbb{R}$

$$h(x) = \inf\{f(y) + g(x-y) \mid y \in \mathbb{R}\} = (f \otimes g)(x)$$

If there exist two real numbers s_0 and r_0 such that for all $x \in \mathbb{R}$

$$f(x) \geq s_0 x - r_0 \qquad g(x) \geq s_0 x - r_0$$

then h is convex.

Remark 3.6 For *convex* $f, g \in \mathcal{F}$, we can choose $s_0 = r_0 = 0$ in the claim, since we have $f(x) \geq 0$ and $g(x) \geq 0$. So the min–plus convolution of convex $f, g \in \mathcal{F}$ needs to be convex.

Remark 3.7 We denote by

$$\operatorname{Conv} \mathbb{R} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is convex}\}$$

the set of all convex functions from the real numbers to the real numbers.

Definition 3.8 We say that $f \in \operatorname{Conv} \mathbb{R}$ is *closed*, or lower semi–continuous, if

$$\liminf_{x \to x_0} f(x) \geq f(x_0) \qquad \forall\, x_0 \in \mathbb{R}$$

The set of all closed convex functions is denoted by $\overline{\operatorname{Conv}}\, \mathbb{R}$.
Geometrically this closedness property can be described as follows:

Proposition 3.9 The function f is closed if and only if one of the following conditions hold:

1. $\operatorname{epi} f$ is a closed subset of $\mathbb{R}^2$.
2. the sublevel–sets
$$S_r(f) := \{x \in \mathbb{R} \mid f(x) \leq r\}$$
are closed intervals of $\mathbb{R}$ (possibly empty), for all $r \in \mathbb{R}$.

It is useful to know which operations preserve closedness. Therefore consider the following three propositions.

Proposition 3.10 Let $f_1, \ldots, f_m$ be m closed convex functions and $t_1, \ldots, t_m$ be positive numbers. If there exists $x_0 \in \mathbb{R}$, such that $f_j(x_0) < \infty$ for $j = 1, \ldots, m$, then the function

$$f := \sum_{i=1}^{m} t_i f_i$$

is in $\overline{\operatorname{Conv}}\, \mathbb{R}$.

Proposition 3.11 Let $\{f_j\}_{j\in J}$ be a family of closed convex functions. If there exists $x_0 \in \mathbb{R}$ such that $\sup_{j\in J} f_j(x_0) < \infty$ then the function

$$f := \sup_{j\in J} f_j$$

is in $\overline{\operatorname{Conv}}\,\mathbb{R}$.

For our purposes the most valuable proposition is the following one.

Proposition 3.12 Let $f, g \in \overline{\operatorname{Conv}}\,\mathbb{R}$. Then the function

$$h := f \otimes g$$

is in $\overline{\operatorname{Conv}}\,\mathbb{R}$.

3.2 Conjugate Functions

Definition 3.13 Let f be a function satisfying

1. $f \not\equiv +\infty$
2. $\exists\, m, b$ such that $f(x) \geq mx + b$ for all $x \in \mathbb{R}$.

Then we define the *conjugate function* of f by

$$f^*(s) = \sup\{sx - f(x) \mid x \in dom\, f\}$$

The transformation $f \mapsto f^*$ is know as the *Fenchel correspondence* or *Fenchel-Transform.*

Remark 3.14 For functions $f \in \mathcal{F}$ the two conditions of the former definition hold true:

- We have $f(x) = 0$ for $x < 0$.
- Taking $m = 0, b = 0$ yields $f(x) \geq 0$.

Definition 3.15 We define the *biconjugate function* of f by

$$f^{**}(x) = \sup\{sx - f^*(s) \mid s \in dom\, f^*\}$$

Proposition 3.16 Let $f \in \operatorname{Conv}\mathbb{R}$. Then

1. the conjugate of f is a closed convex function, that is $f^* \in \overline{\operatorname{Conv}}\,\mathbb{R}$.
2. the biconjugate of f is its closure, that is $f^{**} = \mathrm{cl} f$.

So for a **closed** convex function f we have $f = f^{**}$.
The next proposition could be the key for computing the min–plus convolution.

Proposition 3.17 Let $f_1, f_2 \in \overline{\text{Conv}}\,\mathbb{R}$, minorzed by a common affine function. Then

$$(f_1 \otimes f_2)^* = f_1^* + f_2^*$$

Proof The proof illustrates some properties of extremization. For $s \in \mathbb{R}$, we have

$$\begin{aligned}
(f_1 \otimes f_2)^*(s) &= \sup_x \{sx - \inf_{x=x_1+x_2} f_1(x_1) + f_2(x_2)\} \\
&= \sup_{x=x_1+x_2} \{s(x_1+x_2) - f_1(x_1) - f_2(x_2)\} \\
&= \sup_{x_1,x_2} \{s(x_1+x_2) - f_1(x_1) - f_2(x_2)\} \\
&= \sup_{x_1} \{sx_1 - f_1(x_1)\} + \sup_{x_2} \{sx_2 - f_2(x_2)\}
\end{aligned}$$

and we recognize $f_1^*(s) + f_2^*(s)$ in this last expression. □

So **if** f_1 and f_2 are both closed convex functions, we know by proposition 3.12 that $f_1 \otimes f_2$ is closed convex and therefore we know $f_1 \otimes f_2 = (f_1 \otimes f_2)^{**}$. Summarizing we have

$$f_1 \otimes f_2 = (f_1 \otimes f_2)^{**} = (f_1^* + f_2^*)^* \tag{3.0}$$

This suggests an alternative way to compute the min–plus convolution. There are two basic question that have to be answered:

1. How hard is the above computation?
   If the functions f_1 and f_2 are not too complicated this can be done relatively easy as will be shown in an example.

2. Can this method be applied to a broader class of functions and if not, can it be modified so that it is applicable to a broader class of functions.

The following is our work and not taken out of [8].
We will answer the first question. Applying the above results, one can for **closed convex functions** compute the min–plus convolution relatively easy for good enough functions, as is shown by the following example.

Example

Consider the rate latency function:

$$f(x) = \begin{cases} 0 & x < T \\ \alpha(x-T) & x \geq T \end{cases}$$

with $T > 0$ and $\alpha > 0$.

Claim 3.18 One obtains for the conjugate

$$f^*(s) = \begin{cases} +\infty & s < 0 \\ s \cdot T & 0 \leq s \leq \alpha \\ +\infty & s > \alpha \end{cases}$$

Proof By definition we have

$$f^*(s) = \sup\{sx - f(x) \mid x \in \mathbb{R}\} = \sup\{\{sx \mid x < T\} \cup \{sx - \alpha x + \alpha T \mid x \geq T\}\}$$

1. $s < 0$: Choose $x_n = -n$, then $s \cdot x_n \to +\infty$ as $n \to \infty$

2. $s = 0$: Obviously $f^*(s) = 0$.

3. $s > 0$:

   a) $\sup\{sx \mid x < T\} = s \cdot T$

   b) $\sup\{sx - \alpha x + \alpha T \mid x \geq T\}$

      i. If $s - \alpha > 0$, then take $x_n = n$ for $n \geq T$. Then $(s - \alpha) \cdot x_n \to \infty$ as $n \to \infty$.

      ii. If $s - \alpha \leq 0$, then the supremum is achieved for $x = T$, yielding $(s - \alpha)T + \alpha T = s \cdot T$

□

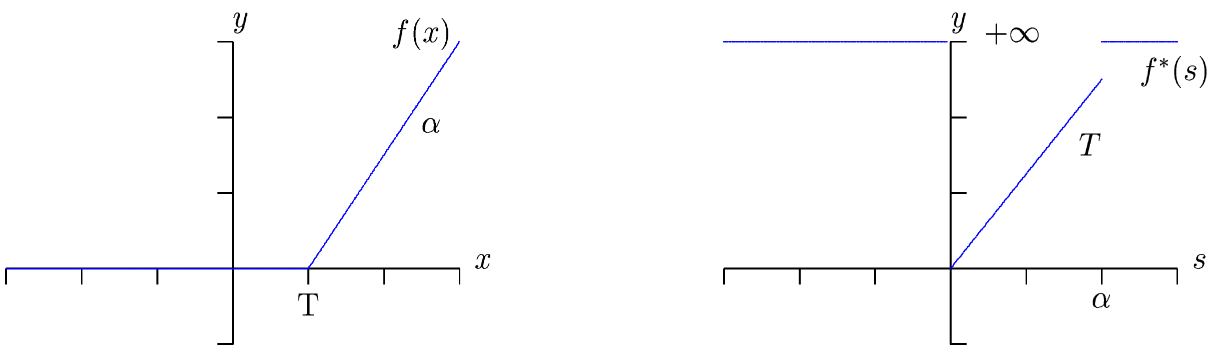

Figure 3.1: Rate Latency Curve and its Conjugate

Now we can compute $f^{**}(t) = \sup\{tx - f^*(x) \mid x \in dom\, f^*\}$. We find

$$f^{**}(t) = \begin{cases} 0 & t < T \\ \alpha(t - T) & t \geq T \end{cases}$$

Proof We have

$$\begin{aligned} f^{**}(t) &= \sup\{tx - f^*(x) \mid x \in dom\, f^*\} \\ &= \sup\{tx - T \cdot x \mid 0 \leq x \leq \alpha\} \end{aligned}$$

1. If $t \geq T$, then $t - T \geq 0$, thus $\sup\{tx - T \cdot x \mid 0 \leq x \leq \alpha\} = (t-T)\alpha$
2. IF $t < T$, then $t - T < 0$, thus $\sup\{tx - T \cdot x \mid 0 \leq x \leq \alpha\} = 0$

□

Thus we have $f^{**} = f$, if f is a rate latency curve.

Remark 3.19 Of course this is not surprising, since a rate latency curve is closed convex. If we would have got a different result the computation would have been wrong.

As an immediate consequence of the above, we can investigate the following situation: Let

$$f_1(x) = \begin{cases} 0 & x < T_1 \\ \alpha(x - T_1) & x \geq T_1 \end{cases} \qquad f_2(x) = \begin{cases} 0 & x < T_2 \\ \beta(x - T_2) & x \geq T_2 \end{cases}$$

be two rate latency curves with $T_2 > T_1 > 0$ and $\beta > \alpha > 0$. Then we have for their conjugates:

$$f_1^*(s) = \begin{cases} +\infty & s < 0 \\ s \cdot T_1 & 0 \leq s \leq \alpha \\ +\infty & s > \alpha \end{cases}$$

and

$$f_2^*(s) = \begin{cases} +\infty & s < 0 \\ s \cdot T_2 & 0 \leq s \leq \beta \\ +\infty & s > \beta \end{cases}$$

So we have for the sum of these two conjugates:

$$f_1^*(s) + f_2^*(s) = \begin{cases} +\infty & s < 0 \\ s \cdot (T_1 + T_2) & 0 \leq s \leq \alpha \\ +\infty & s > \alpha \end{cases}$$

Since $f_1 \otimes f_2$ is **closed** convex by Proposition 3.12, we have:

$$\begin{aligned} (f_1 \otimes f_2)(s) &= ((f_1 \otimes f_2)^{**})(s) && (3.1) \\ &= ((f_1 \otimes f_2)^*)^*(s) && (3.2) \\ &= (f_1^* + f_2^*)^*(s) && (3.3) \\ &= \begin{cases} (s - (T_1 + T_2)) \cdot \alpha & s \geq T_1 + T_2 \\ 0 & s < T_1 + T_2 \end{cases} && (3.4) \end{aligned}$$

Equation (3.2) follows from the definition of the double conjugate. Equation (3.3) uses Proposition 3.17 and (3.4) follows by a computation similar to the one that was done when computing f^{**} above for the rate latency curve. Obviously this is a relatively easy way to determine the min–plus convolution for two functions.
It remains to be seen if we can extend this sort of argument to a broader class of functions.

3.3 Conjugate of Convex and Nonconvex Functions

The next aim was to understand the $*$ operation more fully geometrically. For a rate latency curve f we see what happens: the slope of f is the boarder of $dom\, f^*$ and the right boarder of $\operatorname{supp} f$ becomes the slope of f^*.
So we started computing the conjugate for diverse functions, trying to establish a more general rule.

Burst delay function

The burst delay function is given by

$$\delta_T(t) = \begin{cases} 0 & t < T \\ +\infty & t \geq T \end{cases}$$

Remark 3.20 $\delta_0(t)$ is the neutral element concerning the min–plus convolution.

We can compute the conjugate and derive

$$\delta_T^*(s) = \begin{cases} s \cdot T & s \geq 0 \\ +\infty & s < 0 \end{cases}$$

Proof By definition we have

$$\delta_T^*(s) = \sup_{x \in \mathbb{R}} \{s \cdot x - \delta_T(s)\}$$

1. $s < 0$
   We then take $x_n = -n, n \in \mathbb{N}$ and have $\delta_T(-n) = 0$ for n large enough. So we have $\delta_T^*(s) = +\infty$.

2. $s \geq 0$
   The function $s \cdot x$ is strictly monotone increasing in x in this case, so for $x_n = T - \frac{1}{n}$ we have $\delta_T(x_n) = 0$ and $s \cdot x_n = s \cdot T - s\frac{1}{n}$; letting n tend to infinity yields the assertion.

□

For the biconjugate we derive

$$\delta_T^{**} = \delta_T$$

Proof By defintion we have

$$\delta_T^{**}(t) = \sup\{t \cdot x - \delta_T^*(x) \mid x \in dom(\delta_T^*)\}$$

We have $dom(\delta_T^*) = \mathbb{R}_0^+$.

1. $t - T \geq 0$
   Take $x_n = n, n \in \mathbb{N}$, then $(t - T)x_n \to \infty$ as $n \to \infty$.

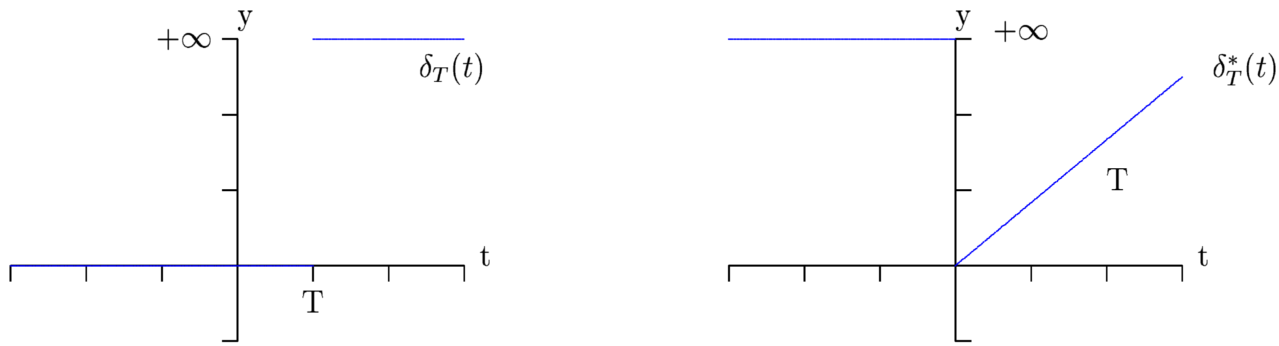

Figure 3.2: Burst Delay Curve and its Conjugate

2. $t - T < 0$
   In this case set $x = 0$ to get the supremum.

□

Obtaining this result is not surprising, since δ_T is closed convex, as one can check easily by looking at the epigraph of δ_T.
δ_0 is the neutral element for the min–plus convolution, when considering only functions $f \in \mathcal{F}$. So we have $f \otimes \delta_0 = f$ for $f \in \mathcal{F}$. So applying our formula for $f \in \mathcal{F}$ we have

$$f^* = (f \otimes \delta_0)^* = f^* + \delta_0^* \qquad (\ddagger)$$

Claim 3.21 For $f \in \mathcal{F}$ and $t < 0$ we have $f^*(t) = +\infty$

Proof Assume the claim would be false. Then we would find a $t_0 < 0$ such that $f^*(t_0) = M < \infty$ and because of the equality ($\ddagger$) $M = M + \delta_0^*(t_0) = \infty$. □

A Nonconvex Function

Consider

$$f(x) = \begin{cases} 0 & x \le 0 \\ x & 0 \le x \le 1 \\ 1 & 1 \le x \le 2 \\ x-1 & 2 \le x \end{cases}$$

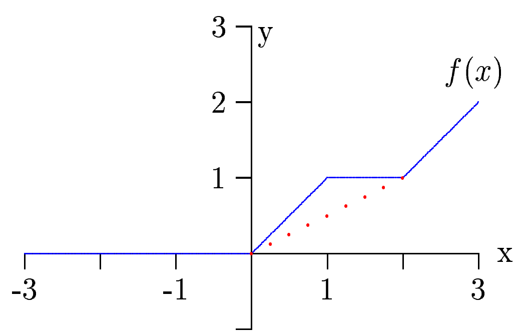

Figure 3.3: Nonconvex Function and its Convex Closure

One easily verifies that this function is neither convex nor concave. But it is bounded from below for example by the affine function $g \equiv 0$, so f^* exists.
One can already realize the convex closure of this function in the graph, which is indicated with dots.

Claim 3.22

$$f^*(s) = \begin{cases} +\infty & s < 0 \\ 0 & 0 \leq s \leq \frac{1}{2} \\ 2s-1 & \frac{1}{2} \leq s \leq 1 \\ +\infty & 1 \leq s \end{cases}$$

Proof Substituting in the definition yields

$$\begin{aligned} f^*(s) &= \sup_{x \in \mathbb{R}} \{sx - f(x)\} \\ &= \max\{A, B, C, D\} \end{aligned}$$

where

$$\begin{aligned} A &:= \sup_{x \leq 0} \{sx\} \\ B &:= \sup_{0 \leq x \leq 1} \{(s-1)x\} \\ C &:= \sup_{1 \leq x \leq 2} \{sx - 1\} \\ D &:= \sup_{x \geq 2} \{(s-1)x + 1\} \end{aligned}$$

1. $s < 0$
   Take $x_n = -n, n \in \mathbb{N}$, then $A \to +\infty$ as $n \to \infty$.

2. $s = 0$
   We have $A = 0$, $B = 0$, $C = -1$ and $D = -1$.

3. $s > 0$
   Here we have $A = 0$.

   a) $s > 1$
      In this case $s - 1 > 0$ and so we take $x_n = n, n \in \mathbb{N}$; then $D \to \infty$ as $n \to \infty$.

   b) $0 < s \leq 1$
      In this case $s - 1 \leq 0$, so $D = (s-1) \cdot 2 + 1 = 2s - 1$. Further we obtain $B = 0$ and $C = 2s - 1$. Since $2s - 1 \geq 0$ if and only if $s \geq \frac{1}{2}$, we have shown the claim.

□

Notice that although f itself is not convex, f^* is in this case!

Claim 3.23

$$f^{**}(s) = \begin{cases} 0 & s \leq 0 \\ \frac{1}{2}s & 0 \leq s \leq 2 \\ s-1 & 2 \leq s \end{cases}$$

Proof We apply the definition for f^{**}

$$\begin{aligned} f^{**}(s) &= \sup\{sx - f^*(x) \mid x \in dom(f^*)\} \\ &= \sup_{0 \leq x \leq 1} \{sx - f^*(x)\} \\ &= \max\{A, B\} \end{aligned}$$

where $A := \sup\{sx \mid 0 \leq x \leq \frac{1}{2}\}$ and $B := \sup\{(s-2)x + 1 \mid \frac{1}{2} \leq x \leq 1\}$.

1. $s < 0$
   In this case $A = 0$ and $B = \frac{1}{2}s$.

2. $s = 0$
   Here we have $A = 0$ and $B = 0$.

3. $s > 0$ Now we get $A = \frac{1}{2}s$. For $s - 2 \leq 0$ we get $B = \frac{1}{2}(s-2) + 1 = \frac{1}{2}s$. For $s - 2 > 0$ we have $B = s - 2 + 1 = s - 1$. We have $s - 1 > \frac{1}{2}s$ if and only if $s > 2$ which proves the claim.

□

For closed convex functions f the inverse of the operation * is * itself, that is one can reconstruct f from f^* by simply taking the conjugate of f^*. But the above example shows that this can **not** be expected for arbitrary functions f. Recall, the reason why we are interested in inverting the * operator is because of the relation

$$(f_1 \otimes f_2)^* = f_1^* + f_2^*$$

that is the * operator converts the min–plus convolution into the addition of two functions. But for computing the min–plus convolution of f_1 and f_2 we have to be able to invert the * operator, that is we are searching for a function κ, such that $\kappa(f^*) = f$. For closed convex f we can simply choose $\kappa(g) = g^*$ but it is not clear how this can be done more general, that is for a broader class of functions.

3.4 Approaches to Construct Convex Functions

Our next idea thus involves constructing convex functions from nonconvex ones, although there is no canonical way of doing this, as far as we know. But the functions under consideration are very easy, actually they are piecewise linear, so maybe we can devise

a method. An example might make the idea clearer.
Consider an affine function given by

$$\gamma_{r,b}(t) = \begin{cases} 0 & t < 0 \\ r \cdot t + b & t \geq 0 \end{cases}$$

Construct the function

$$\tilde{\gamma}_{r,b}(t) = \begin{cases} +\infty & t < 0 \\ r \cdot t + b & t \geq 0 \end{cases}$$

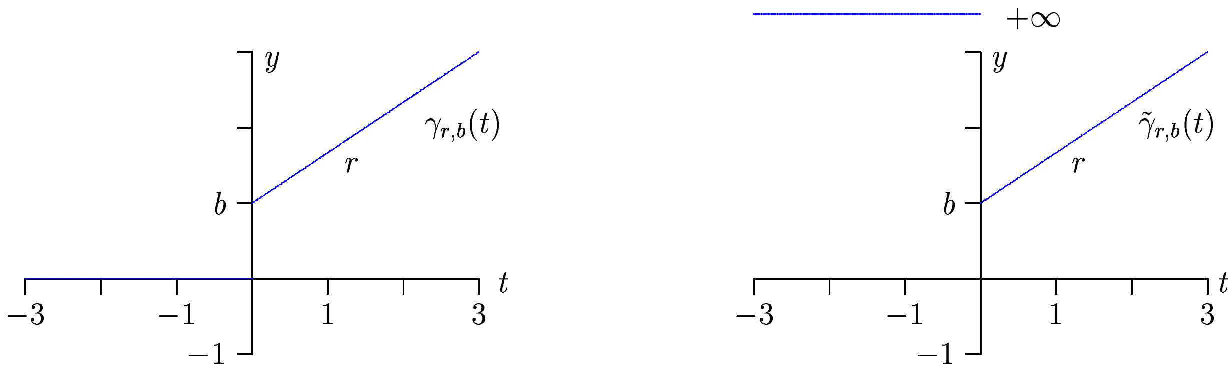

Figure 3.4: Nonconvex Function to Convex Function

Then $\tilde{\gamma}_{r,b}$ is closed convex. Of course one could have transformed $\gamma_{r,b}$ into

$$\hat{\gamma}_{r,b}(t) = \begin{cases} b & t < 0 \\ r \cdot t + b & t \geq 0 \end{cases}$$

for getting another closed convex function. Notice that neither $\tilde{\gamma}_{r,b}$ nor $\hat{\gamma}_{r,b}$ are members of the set $\mathcal{F}$.
The advantage of the constructed closed convex functions is obvious, we have

$$\tilde{\gamma}_{r,b}^{**} = \tilde{\gamma}_{r,b} \quad \text{and} \quad \hat{\gamma}_{r,b}^{**} = \hat{\gamma}_{r,b}$$

But we have a big problem now. We can take a closed convex function f and for example $\tilde{\gamma}_{r,b}$ and compute the min–plus convolution of those functions via

$$f \otimes \tilde{\gamma}_{r,b} = (f \otimes \tilde{\gamma}_{r,b})^{**} = (f^* + \tilde{\gamma}_{r,b}^*)^*$$

But we actually want to compute $f \otimes \gamma_{r,b}$; so we would have to find some relation between $f \otimes \gamma_{r,b}$ and $f \otimes \tilde{\gamma}_{r,b}$.
Let a closed convex f be given, which is already a strong assumption. Following the aforementioned approach and denoting for a function g the constructed closed convex function by $\tilde{g}$ we would have to **find a general relation between $f \otimes g$ and $f \otimes \tilde{g}$!**
This seems to be quite hard and thus this approach is not considered any more by us.

3.5 Noninjectivity of the Conjugacy Operation

Let us look at the function $\gamma_{r,b}$ again. We can compute the conjugate and the biconjugate for this function, although we know that $\gamma_{r,b} \neq \gamma_{r,b}^{**}$.

Claim 3.24

$$\gamma_{r,b}^{*}(s) = \begin{cases} +\infty & s < 0 \\ 0 & 0 \leq s \leq r \\ +\infty & r < s \end{cases}$$

Proof By definition we have

$$\gamma_{r,b}^{*}(s) = \sup_{x \in \mathbb{R}} \{sx - \gamma_{r,b}(x)\}$$

Setting $A := \sup_{x \leq 0}\{sx\}$ and $B := \sup_{x>0}\{(s-r)x - b\}$ we get

$$\gamma_{r,b}^{*}(s) = \max\{A, B\}$$

1. $s < 0$
   Since $\gamma_{r,b} \in \mathcal{F}$ we have $\gamma_{r,b}(s) = +\infty$.

2. $s = 0$
   We have $A = 0$ and since $r, b \geq 0$ we have $B = -b$.

3. $s > 0$
   Here we have $A = 0$. For $s - r > 0$ we have $B = +\infty$. For $s - r \leq 0$ we have $B = -b$. Since $b \geq 0$ we have in this case $\max\{A, B\} = 0$.

□

Claim 3.25

$$\gamma_{r,b}^{**}(t) = \begin{cases} 0 & t \leq 0 \\ r \cdot t & t \geq 0 \end{cases}$$

Proof The definition reads

$$\gamma_{r,b}^{**}(t) = \sup\{tx - \gamma_{r,b}^{*}(x) \mid x \in dom(\gamma_{r,b}^{*})\}$$

So this becomes

$$\gamma_{r,b}^{**}(t) = \sup\{tx \mid 0 \leq x \leq r\}$$

If $t \leq 0$, we set $x = 0$ and for $t > 0$ we set $x = r$. □

Remark 3.26 One can realize a very bad behavior in the transformation * in this case, since the information of b is completely lost in $\gamma_{r,b}^{*}$ and it seems impossible to reconstruct one particular $\gamma_{r,b}$ from a given $\gamma_{r,b}^{*}$. For fixed $r \geq 0$ the family of functions

$$\mathcal{G}_r := \{\gamma_{r,b} \mid b \geq 0\} \subset \mathcal{F}$$

has **the same** conjugate function, namely $\gamma_r^{*} = \gamma_{r,b}^{*}$.

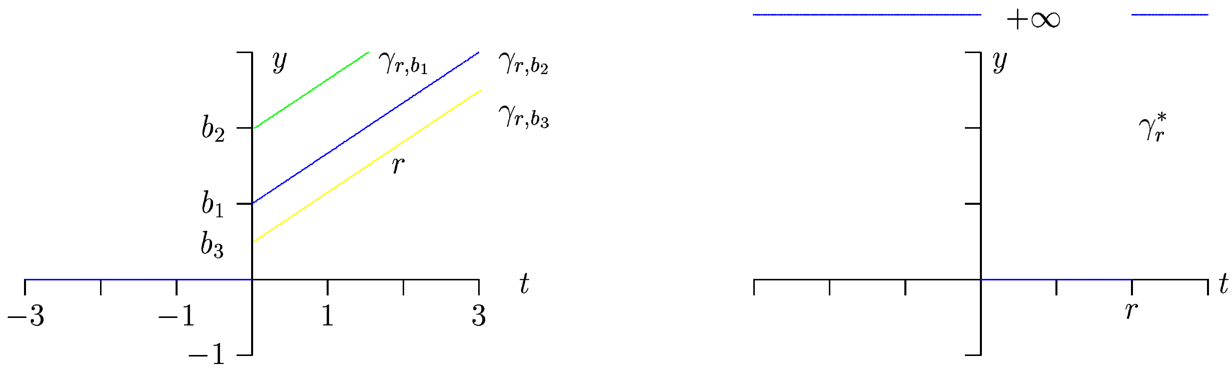

Figure 3.5: Different Functions Having the Same Conjugate

The following picture illustrates the situation graphically: the three functions $\gamma_{r,b_i}, i = 1, 2, 3$ have the same conjugate γ_r^*.

At least for this particular function this phenomenon can be fixed. Consider again the transformation idea, in particular the function

$$\hat{\gamma}_{r,b}(t) = \begin{cases} b & t < 0 \\ r \cdot t + b & t \geq 0 \end{cases}$$

Computing the conjugate yields

$$\hat{\gamma}^*_{r,b}(s) = \begin{cases} +\infty & s < 0 \\ -b & 0 \leq s \leq r \\ +\infty & r < s \end{cases}$$

Proof By definition we have

$$\hat{\gamma}^*_{r,b}(s) = \sup_{x \in \mathbb{R}} \{sx - \hat{\gamma}_{r,b}(x)\}$$

Setting $A := \sup_{x \leq 0}\{sx - b\}$ and $B := \sup_{x > 0}\{(s - r)x - b\}$ we get

$$\gamma^*_{r,b}(s) = \max\{A, B\}$$

1. $s < 0$
   In this case we choose $x_n = -n, n \in \mathbb{N}$ and then $A \to \infty$, as $n \to \infty$.

2. $s = 0$
   We have $A = -b$ and since $r, b \geq 0$ we have $B = -b$.

3. $s > 0$
   Here we have $A = -b$. For $s - r > 0$ we have $B = +\infty$. For $s - r \leq 0$ we have $B = -b$.

□

So here we don't lose the information on b and at least the conjugate of each $\hat{\gamma}_{r,b}$ is unique. So we have at least the chance to find a unique retransformation, yielding $\gamma_{r,b}$ from the information of $\hat{\gamma}_{r,b}$.
The biconjugate of $\hat{\gamma}_{r,b}$ is of course the function itself again; therefore we transformed $\gamma_{r,b}$ into $\hat{\gamma}_{r,b}$, since $\hat{\gamma}_{r,b}$ is a closed convex function.
These considerations make the transformation idea – in spite of the related problems – interesting again.

3.6 Change From Convex to Concave Functions

Consider the function

$$f(x) = \begin{cases} 0 & x < a \\ x - a & a \leq x \leq a+1 \\ m \cdot (x-a) - m + 1 & a+1 \leq x \end{cases}$$

with $m \geq 1$; that amounts to saying f *should be convex.* Then we obtain for the conjugate

$$f^*(s) = \begin{cases} +\infty & s < 0 \\ s \cdot a & 0 \leq s \leq 1 \\ s \cdot (a+1) - 1 & 1 \leq s \leq m \\ +\infty & m < s \end{cases}$$

Proof We set

$$\begin{aligned} A &:= \sup_x \{sx \mid x < a\} \\ B &:= \sup_x \{(s-1)x + a \mid a \leq x \leq a+1\} \\ C &:= \sup_x \{(s-m)x + ma + m - 1 \mid a+1 \leq x\} \end{aligned}$$

Then $f^*(s) = \max\{A, B, C\}$.

1. $s < 0$
   Here we have $A = +\infty$.

2. $s = 0$
   In this case $A = 0$, $B = 0$ and $C = -m \cdot (a+1) + m \cdot (a+1) - 1 = -1$.

3. $s > m$
   We then get $C = +\infty$.

4. $0 \leq s \leq 1$
   In this case we have $s - 1 \leq 0$. So for B we get $B = (s-1) \cdot a + a = s \cdot a$, and for A we have $A = s \cdot a$. By the assumption $m \geq 1$ we get $s - m < 0$, so we have $C = (s-m) \cdot (a+1) + ma + m - 1 = s \cdot (a+1) - 1$. Further we have $s \cdot a \geq s \cdot (a+1) - 1$ if and only if $s \leq 1$.

5. $1 \le s \le m$
   In this case we have $s - 1 \ge 0$, so for B we have $B = (s-1) \cdot (a+1) + a = s \cdot (a+1) - 1 = C$ and $A = s \cdot a$.

□

Now one can ask what happens to the conjugate if we change the parameter m such that f is not convex any more, but still remains in $\mathcal{F}$, that is we choose $0 \le m < 1$. For $0 \le m < 1$ the conjugate of f is given by

$$f^*(s) = \begin{cases} +\infty & s < 0 \\ s \cdot a & 0 \le s \le m \\ +\infty & m < s \end{cases}$$

Proof We set

$$\begin{aligned} A &:= \sup_x \{sx \mid x < a\} \\ B &:= \sup_x \{(s-1)x + a \mid a \le x \le a+1\} \\ C &:= \sup_x \{(s-m)x + ma + m - 1 \mid a+1 \le x\} \end{aligned}$$

Then $f^*(s) = \max\{A, B, C\}$.

1. $s < 0$
   Here we have $A = +\infty$.

2. $s = 0$
   In this case $A = 0$, $B = 0$ and $C = -m \cdot (a+1) + \cdot m(a+1) - 1 = -1$.

3. $s > m$
   We then get $C = +\infty$.

4. $0 \le s \le m$
   In this case we have by the assumption $m < 1$ $s - 1 < s - m \le 0$. So for B we get $B = (s-1) \cdot a + a = s \cdot a$, and for A we have $A = s \cdot a$. Since $s - m < 0$, we have $C = (s-m) \cdot (a+1) + ma + m - 1 = s \cdot (a+1) - 1$. Further we have $s \cdot a \ge s \cdot (a+1) - 1$ if and only if $s \le 1$.

□

So as it seems one loses the information about the slope $a+1$ which cannot be reconstructed that easily. In the following picture $c := m \cdot (a+1) - 1 \ge a$ if and only if $m \ge 1$.

If we take $m = 0$ the conjugate function becomes

$$f^*(s) = \begin{cases} +\infty & s < 0 \\ 0 & s = 0 \\ +\infty & s > 0 \end{cases}$$

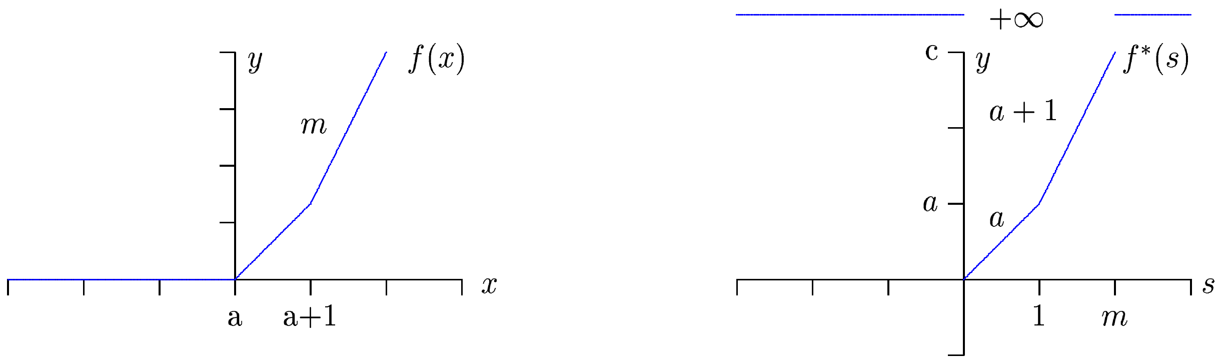

Figure 3.6: Conjugate in the Convex Case, i.e. $m \geq 1$

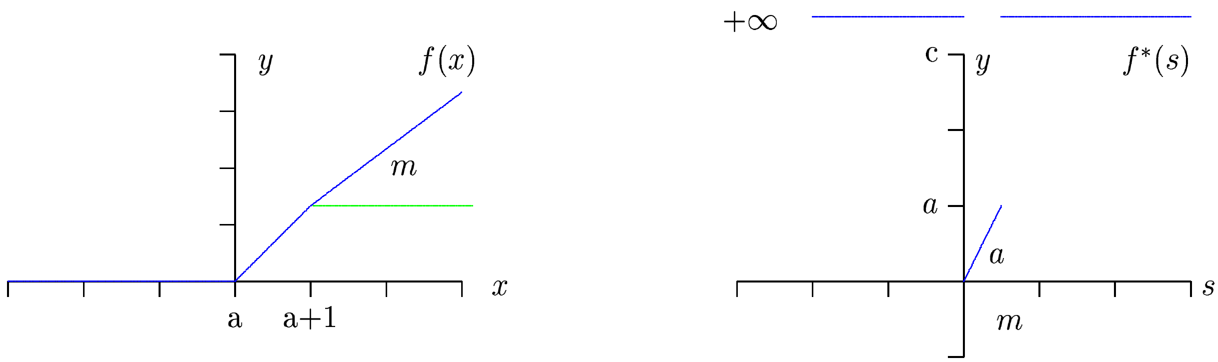

Figure 3.7: Conjugate in the Nonconvex Case, i.e. $0 \leq m < 1$

So all information on the parameter a is completely lost and there is only little chance to reconstruct f only from its conjugate!
But the function f with $m = 0$ is praxis–relevant since it describes a flow with a quota and can thus not be discarded.
A possible solution is in this case to consider $-f$, but this still is *not* closed convex; furthermore one has to have information on how this changes the min–plus convolution, that is how $f \otimes g$ and $-f \otimes g$ relate to one another.

Another conclusion can be drawn from this example: Continuity, a basic and usually helpful quality of a function, does not influence the behavior of the operation *, since the choice of the parameter m does not influence the continuity of f. Of course f is (strongly) differentiable only in the case $m = 1$ and since (at least in $\mathbb{R}$) diffentiability implies continuity, this quality is also irrelevant to the behavior of the conjugacy operation!
We have already encountered that the * operator is not injective if its domain is $\mathcal{F}$: consider the discussion in the example of $\gamma_{r,b}$. So there is no chance of getting a function

$$\kappa : ran(^*) \to \mathcal{F}, \qquad \kappa(f^*) = f$$

For example $|\kappa(\gamma^*_{r,b})| > 1$; with the notation used earlier we have for fixed r

$$\kappa(\gamma^*_{r,b}) = \mathcal{G}_r.$$

So in general the values of κ range through the powerset of $\mathcal{F}$.
Let

$$\mathcal{C} := \{f : \mathbb{R} \to \mathbb{R} \mid f^{**} = f\}$$

Then $\mathcal{C}$ is nonempty. Changing the domain of the operation $*$ from $\mathcal{F}$ to $\mathcal{F} \cap \mathcal{C}$, we find a unique inverse κ, namely $\kappa = *$.

3.7 Geometric Approach of Interpreting Conjugacy

In general one can realize in the linear case that changes to the slope (which is then constant!) of the function f effect the domain of the conjugate f^*, whereas changes in the domain of f, such as moving the paramter a, translate in the slope of the conjugate f^*.
For deriving further insight in this behavior we investigate for $n \in \mathbb{N}$ the functions

$$g_n(x) = \begin{cases} 0 & x < 0 \\ x^{2n} & x \geq 0 \end{cases}$$

These functions are all closed convex, as one can realize by drawing the graph, and belong to $\mathcal{F}$; so they are the best functions we can have for our transformation.
We want to compute the conjugate to see how the (now changing) slope is reflected in the conjugate. By definition we have

$$g_n^*(s) = \sup_{x \in \mathbb{R}} \{sx - g_n(x)\}$$

So for $s < 0$ we get $g_n^*(x) = +\infty$. For $s \geq 0$ we have to compute

$$\max(0, \sup_{x \geq 0} \{sx - x^{2n}\})$$

By differentiation with respect to x and setting the result equal zero, we get

$$x_{max} = \left(\frac{s}{2n}\right)^{\frac{1}{2n-1}} \geq 0$$

This has to be the maximum, since the second derivative is negative. Substituting x_{max} we get for the value of

$$\begin{aligned} g_n^*(s) &= s \cdot x_{max} - x_{max}^{2n} \\ &= s \cdot \left(\frac{s}{2n}\right)^{\frac{1}{2n-1}} - \left(\frac{s}{2n}\right)^{\frac{2n}{2n-1}} \\ &= 2n \frac{s}{2n} \cdot \left(\frac{s}{2n}\right)^{\frac{1}{2n-1}} - \frac{s}{2n} \cdot \left(\frac{s}{2n}\right)^{\frac{1}{2n-1}} \\ &= \frac{2n-1}{2n} \cdot s \cdot \left(\frac{s}{2n}\right)^{\frac{1}{2n-1}} \\ &= (2n-1) \cdot \left(\frac{s}{2n}\right)^{\frac{2n}{2n-1}} \geq 0 \end{aligned}$$

So we have

$$g_n^*(s) = \begin{cases} +\infty & s < 0 \\ (2n-1) \cdot \left(\frac{s}{2n}\right)^{\frac{2n}{2n-1}} & s \geq 0 \end{cases}$$

One observes that the slope of x^{2n} does not influence the domain of $g_n^*(s)$, so the statement:
"Slopes of a function f influence the domain of f^*" is false in general, although it seems to be valid for piecewise linear functions.

3.8 Nonlinearity of the Conjugacy Operation

Another desirable property of the transformation * is linearity. But * does not posess this property in general.
Proof We consider the operation $^* : (\mathcal{F}, \oplus, \otimes) \to (\mathcal{G}, \tilde{\oplus}, \tilde{\otimes})$, where $\oplus = \min$ and $\otimes = +$, since we are dealing with the min–plus convolution. The set $\mathcal{G} \supset ran(^*)$ is endowed with two operations $\tilde{\oplus}$ and $\tilde{\otimes}$.
Let $f, g \in \mathcal{F}$, $k \in \mathbb{R}$ be given. For a linear transformation $\mathcal{M}$ between $\mathcal{F}$ and $\mathcal{G}$ the following has to hold:

$$\mathcal{M}(k \otimes f) = k \tilde{\otimes} \mathcal{M}(f) \tag{3.5}$$
$$\mathcal{M}(f \oplus g) = \mathcal{M}(f) \tilde{\oplus} \mathcal{M}(g) \tag{3.6}$$

For the operation * we have

$$\begin{aligned} (k \otimes f)^*(s) &= \sup_{x \in \mathbb{R}} \{sx - (k + f(x))\} \\ &= \sup_{x \in \mathbb{R}} \{sx - f(x)\} - k \\ &= k \tilde{\otimes} f^*(s) \end{aligned}$$

if we set

$$a \tilde{\otimes} b := b - a$$

We also require

$$\min(f, g)^* = f^* \tilde{\oplus} g^*$$

According to [8], we have

$$\min(f, g)^* = \sup(f^*, g^*)$$

So setting

$$f \tilde{\oplus} g = \sup(f^*, g^*)$$

we satisfy the linearity conditions. □
Choosing the operations $\tilde{\oplus}$ or $\tilde{\otimes}$ different results in the loss of the linearity of the Fenchel–Transform.

3.9 Splitting Into Convex Functions

Since the Fenchel–Transformation works quite well for convex functions we try to split non–convex functions into convex components; although this approach will probably not work in general, for piecewise linear functions it might present a suitable alternative. Let $T_1 < T_2, \alpha < \beta \in \mathbb{R}_+$ be given and consider the following function

$$g(t) = \begin{cases} 0 & t \leq T_2 \\ \beta \cdot (t - T_2) & T_2 \leq t \leq T_3 \\ \alpha \cdot (t - T_1) & t \geq T_3 \end{cases}$$

where $T_3 = \frac{\beta T_2 - \alpha T_1}{\beta - \alpha}$ denotes the intersection of the two functions

$$f_{T_1}(t) = \begin{cases} 0 & t \leq T_1 \\ \alpha \cdot (t - T_1) & t \geq T_1 \end{cases} \qquad f_{T_2}(t) = \begin{cases} 0 & t \leq T_2 \\ \beta \cdot (t - T_2) & t \geq T_2 \end{cases}$$

Then we have $g(t) = \min\{f_{T_1}(t), f_{T_2}(t)\}$.

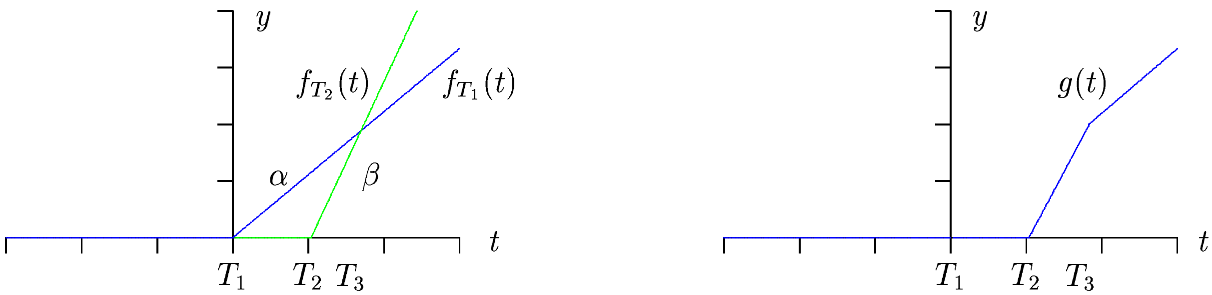

Figure 3.8: Splitting Into Convex functions

Now define

$$\eta(g(t)) = \eta(\min\{f_{T_1}(t), f_{T_2}(t)\}) := \min\{f^*_{T_1}(t), f^*_{T_2}(t)\}$$

For the conjugates of $f_{T_1}(t)$ and $f_{T_2}(t)$ we have

$$f^*_{T_1}(t) = \begin{cases} +\infty & t < 0 \\ t \cdot T_1 & 0 \leq t \leq \alpha \\ +\infty & t > \alpha \end{cases}$$

and

$$f^*_{T_2}(t) = \begin{cases} +\infty & t < 0 \\ t \cdot T_2 & 0 \leq t \leq \beta \\ +\infty & t > \beta \end{cases}$$

Since we assumed $\beta > \alpha$ and $T_1 < T_2$ we get

$$\eta(g(t)) = \begin{cases} +\infty & t < 0 \\ t \cdot T_1 & 0 \leq t \leq \alpha \\ t \cdot T_2 & \alpha < t \leq \beta \\ +\infty & t > \beta \end{cases}$$

Notice that all four parameters are contained in the image of g under η. If we had transformed the function g using the * operator, we would have lost some information. Recall that usually we have

$$g(t)^* = \min\{f_{T_1}(t), f_{T_2}(t)\}^* = \sup\{f^*_{T_1}(t), f^*_{T_2}(t)\}$$

The loss of information can be prevented with the transformation η.
We actually want to compute the min–plus convolution of the two functions $f_{T_1}(t)$ and $f_{T_2}(t)$. We have shown in (3.1) to (3.4)

$$(f_{T_1} \otimes f_{T_2})(s) = \begin{cases} (s - (T_1 + T_2)) \cdot \alpha & s \geq T_1 + T_2 \\ 0 & s < T_1 + T_2 \end{cases}$$

Now one has to answer the question whether one can find a general way for obtaining the min–plus convolution from the knowledge of η; at least there is a fair chance, since we have all four parameters available.
The definition of η presents a potential problem; if we can not write the function g as the minimum of two other functions, then η is undefined.
We think we would have to think of a way how to define, for example $\eta(f \otimes g)$. But then we are back at the original problem of finding a transformation so that $\eta(f \otimes g) = \eta(f) \odot \eta(g)$, where $\odot$ is some operation that is easy to compute.
Furthermore from a mathematical point of view all these approaches are not beautiful because there seems to be no unifying theory behind the problem. There are too many special cases in which the approaches we have tried do not work.

3.10 A More General Approach

One can look again at the Laplace–Transformation; we are trying to find a similar transformation simplifying the computation of the min–plus convolution. Recall that the Laplace–Transformation for a given function f reads

$$\mathcal{L}(f)(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\, dt$$

The conventional *convolution* of two functions f, g is given by

$$(f \otimes g)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t - \tau)\, d\tau$$

Now the Laplace transformation yields

$$\mathcal{L}(f \otimes g)(s) = \mathcal{L}(f)(s) \cdot \mathcal{L}(g)(s)$$

We would like to devise a similar transform with a similar quality for the min–plus convolution.

Investigating the situation heuristically (that is, not mathematically exact), we can derive the following

$$\begin{aligned} \mathcal{L}(f)(s) &= \int_{-\infty}^{\infty} f(t)e^{-st}\, dt \\ &\approx \sum_{t=-\infty}^{\infty} f(t)e^{-st} \\ &= \bigoplus_{t=-\infty}^{\infty} f(t) \odot e^{-st} \end{aligned}$$

This consideration yields the following approach:

$$\kappa_g(f)(s) = \bigoplus_{t=-\infty}^{\infty} f(t) \odot g(s,t)$$

Of course there are various parameters to be specified:

- The $\oplus$ has to be filled with a meaning.
- The $\odot$ has to be filled with a meaning.
- The suitable choice of the function $g(s,t)$ is of utmost importance.

One can realize that this general approach includes (more or less) the Fenchel–Transformation, if one chooses in particular

$$\begin{aligned} g(s,t) &:= s \cdot t \\ a \odot b &:= b - a \\ \bigoplus &:= \sup \end{aligned}$$

We then have

$$\kappa_g(f)(s) = \sup_{t \in \mathbb{Z}} \{s \cdot t - f(t)\}$$

If we let t range through $\mathbb{R}$ instead of $\mathbb{Z}$ this is precisely the definition of the conjugate of f.

Maybe if we choose the function g different from the above for nonconvex functions we could get a transformation that posesses the quality that we want to have.
From a mathematical point of view this approach should be preferred because it contains the original conjugacy operation as a special case and we try to extend a known theory.

Let us investigate some properties of the above defined κ_g we would like to have. First of all we can look at how the transformation influences the min–plus convolution. Let therefore $f, h \in \mathcal{F}$ be given

$$\begin{aligned} \kappa_g(f \otimes h)(s) &= \bigoplus_{t=-\infty}^{\infty} (f \otimes h)(t) \odot g(s,t) \\ &\stackrel{!}{=} \left(\bigoplus_{t=-\infty}^{\infty} f(t) \odot g(s,t) \right) \tilde{\bigotimes} \left(\bigoplus_{t=-\infty}^{\infty} h(t) \odot g(s,t) \right) \qquad (3.7) \\ &= \kappa_g(f)(s) \tilde{\bigotimes} \kappa_g(h)(s) \end{aligned}$$

1. If one chooses for the operator $\bigoplus$ the usual $+$, for example, then one has the problem of having to compute an infinite sum, which usually is not an easy task. In this case choosing $\odot$ or g in a way that the sum is finite is desirable.

2. In order for (3.7) to hold the operator $\odot$ probably has to satisfy an equation of the form
$$h(t-j) \odot g(s,t) = h(t) \odot g(s,t+j)$$
since $(f \otimes h)(t) = \min_{0 \leq j \leq t}\{f(j) + h(t-j)\}$ and in equation (3.7) $h(t-j)$ does not occur any more.

3. Probably the choice of $\tilde{\bigotimes}$ should involve the function min, since this appears in the min–plus convolution and in order to make (3.7) hold we should not lose the min completely. On the other hand the operation $\tilde{\bigotimes}$ should be easy to compute, so this is somewhat contradictory to choosing an operation involving the minimum.

4. With this approach we probably can use some ideas presented in [8], for example one could try to find eigenfunctions, that is special functions g.

4 Computing the Min–Plus Convolution

The following computations were made to investigate the behavior of the min–plus convolution with nonconvex (but practically relevant) functions. It will become clear why one is interested in a more general theory of computing the min–plus convolution, since the computations involved are rather tedious and somewhat technical.
The aim is to be able to develop a technique that enables us to forecast the affine functions encountered in the proofs of the assertions below beforehand. One will see that these affine functions depend solely on the parameters of the piecewise linear functions; the min–plus convolution at a given point t is just the minimum of these affine functions evaluated at t.
So if one can predict these affine functions reliably, one can easily compute the min–plus convolution, either by hand or with a computer.

4.1 Rate Latency Curve and a Nonconvex Function With 2 Slopes

We take into account the following special two functions

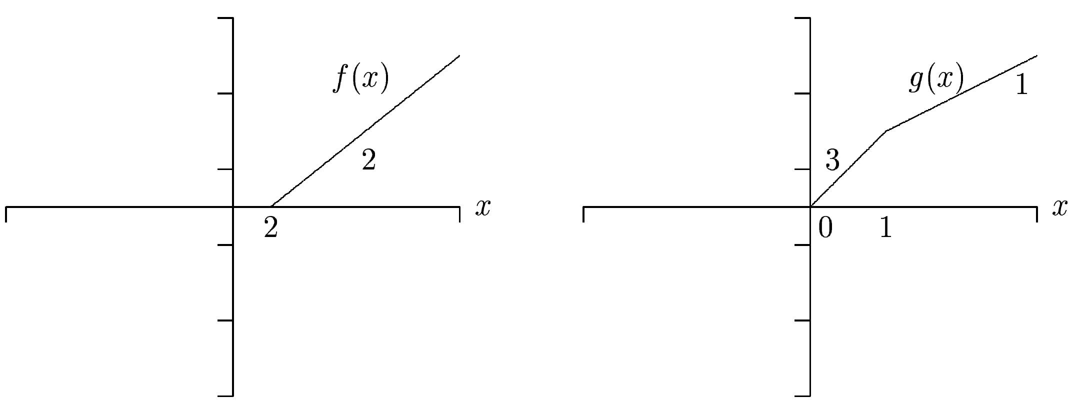

Figure 4.1: Rate Latency Curve and Nonconvex Function

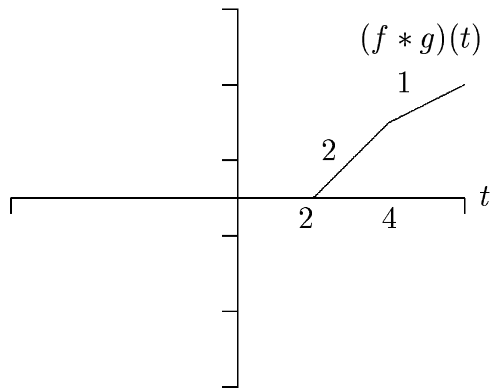

Figure 4.2: Min–plus Convolution of RLC and Nonconvex Function

The functions' formal definition is:

$$f(x) = \begin{cases} 0 & x < 2 \\ 2 \cdot (x-2) & x \geq 2 \end{cases}$$

and

$$g(x) = \begin{cases} 0 & x < 0 \\ 3 \cdot x & 0 \leq x \leq 1 \\ (x-1)+3 & 1 \leq x \end{cases}$$

We can compute their min–plus convolution. We get

$$(f * g)(t) = \begin{cases} 0 & t < 2 \\ 2 \cdot (t-2) & 2 \leq t \leq 4 \\ t & 4 \leq t \end{cases}$$

We do not provide a proof of this particular claim here, since a more general formula will be proven below. The interested reader can simply check the assumption by substituting f and g in the definition for $f * g$.

The parameter 3, i.e. the slope of g on the interval $[0, 1]$, does not appear in this last expression, so maybe the min–plus convolution does not depend on all parameters. This would be very helpful since one could "forget" respectively "lose" some information in a transform without any harm!

4.2 RLC and Nonconvex Piecewise Linear Function With 2 Slopes

Motivated by this observation the following more rigid and general considerations were taken. We investigate the min–plus convolution of the parameterized families of function f and g. Their graphs and the corresponding min–plus convolutions are depicted on the next page.

The functions' formal definition is:

$$f(x) = \begin{cases} 0 & x < T \\ m \cdot (x - T) & x \geq T \end{cases}$$

and

$$g(x) = \begin{cases} 0 & x < A \\ a \cdot (x - A) & A \leq x \leq B \\ b \cdot (x - B) + a \cdot (B - A) & B \leq x \end{cases}$$

In the following we assume $a \geq b$ (else the function g would be convex and then we could apply previous results) and $A < B$. As indicated above we are only interested in the case $A, T > 0$.

We want to compute the following quantity for all $t \in \mathbb{R}$:

$$(f * g)(t) = \inf_{0 \leq s \leq t} \{f(t - s) + g(s)\}$$

that is the min–plus convolution of f and g. Since $f, g : \mathbb{R} \to \mathbb{R}_0^+$, we clearly have $f * g \geq 0$.

Below we will show, that we get the following expressions for the min–plus convolution, assuming $T, A > 0$, $A < B$ and $a \geq b$:

1. Case $a \geq b \geq m$

$$(f * g)(t) = \begin{cases} 0 & t \leq T + A \\ m \cdot (t - (T + A)) & T + A \leq t \end{cases}$$

2. Case $b \leq a \leq m$

$$(f * g)(t) = \begin{cases} 0 & t \leq T + A \\ a \cdot (t - (T + A)) & T + A \leq t \leq T + B \\ b \cdot (t - (T + B)) + a \cdot (B - A) & T + B \leq t \end{cases}$$

3. Case $b \leq m \leq a$

$$(f * g)(t) = \begin{cases} 0 & t \leq T + A \\ m \cdot (t - (T + A)) & T + A \leq t \leq T + D \\ b \cdot (t - (T + B)) + a \cdot (B - A) & T + D \leq t \end{cases}$$

where D is defined as

$$D = \frac{A \cdot (m - a) + B(a - b)}{m - b}$$

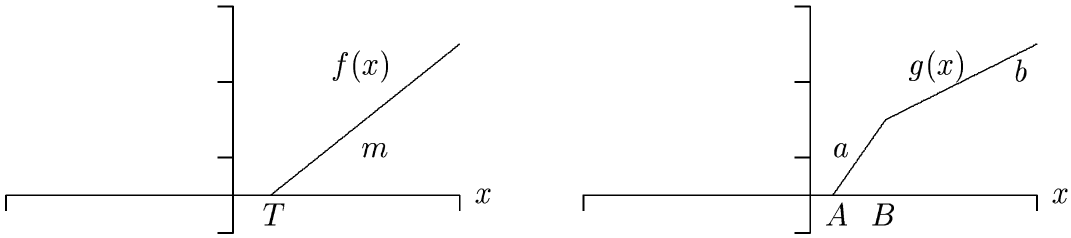

Figure 4.3: RLC and Nonconvex Piecewise Linear (NPL) Function

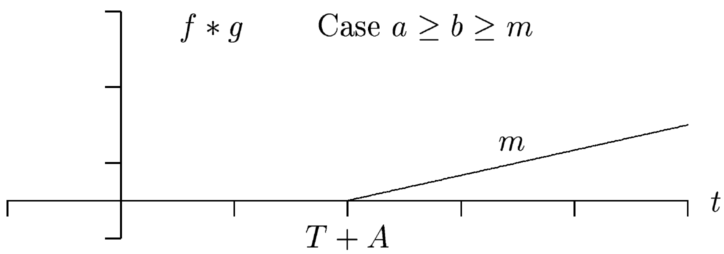

Figure 4.4: Min–plus Convolution of RLC and Nonconvex Piecewise Linear Function

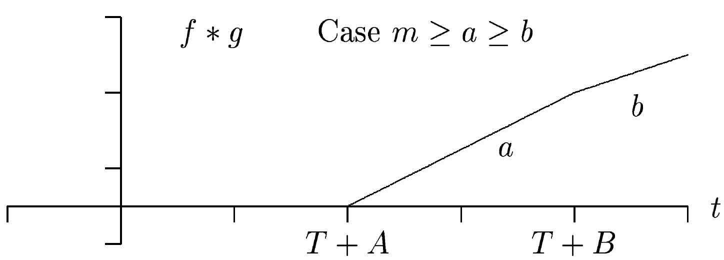

Figure 4.5: Min–plus Convolution of RLC and Nonconvex Piecewise Linear Function

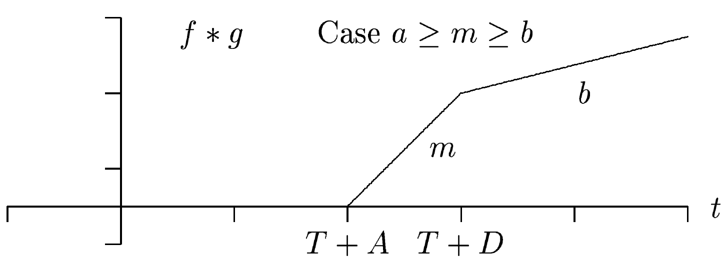

Figure 4.6: Min–plus Convolution of RLC and Nonconvex Piecewise Linear Function

Remark 4.1 Notice that the interval boundaries, on which $f * g$ is piecewise linear, depend on the parameters in a nontrivial fashion. Hence it seems highly unlikely that one can find a general transform that in certain cases "loses" the right parameters in order to compute the correct min–plus convolution.

Remark 4.2 The proofs in this section have a certain structure. In a first step, which will be referred as *STEP I* below, we divide the computation of $f(t-s)+g(s)$ in a certain way, making it more tractable. We will get expressions (essentially piecewise linear functions) we can work with easier in the second step, which will be referred to as *STEP II*. In STEP II we compute the min–plus convolution, which is the pointwise minimum of the piecewise linear functions derived in STEP I. This minimum heavily depends on the relations holding for the parameters a, b and m, so we will have to distinguish various cases. In our computation of the min–plus convolution for other functions we will proceed in a very similar manner.

Remark 4.3 At the end of STEP I we basically have all the information we need to compute the min–plus convolution. The rather tedious computations in STEP II can be done by a computer, since they simply involve computing the minimum at every point $t > 0$ of a finite set of piecewise linear functions. So if we find a way to predict these functions reliably, we would have an easy way to compute the min–plus convolution by means of a computer program! This is in particular of interest for the application in a network, since running such a program on a router would enable us e.g. to give worst case estimates on delays, thus ensuring a better QoS.

Proof

STEP I

We wish to compute

$$(f * g)(t) = \inf_{0 \le s \le t} \{f(t-s) + g(s)\}$$

As already mentioned, we have $f * g \ge 0$ in our case. We will compute $f * g$ in steps. We note that $t - s < T$ iff $s > t - T$.

Step 1: $t - T \le A$

We have

$$\begin{aligned}
(f * g)(t) &= \inf_{0 \le s \le t} \{f(t-s) + g(s)\} \\
&= \min\{\inf_{0 \le s \le t-T} \{f(t-s) + g(s)\}, \inf_{t-T \le s \le t} \{f(t-s) + g(s)\}\} \\
&= \min\{\inf_{0 \le s \le t-T} \{f(t-s) + g(s)\}, \inf_{t-T \le s \le t} \{g(s)\}\} \\
&= \min\{\inf_{0 \le s \le t-T} \{f(t-s) + g(s)\}, 0\} \\
&= 0
\end{aligned}$$

Step 2: $A \le t - T \le B$

As before we have

$$(f * g)(t) = \min\{\inf_{0\le s\le t-T}\{f(t-s)+g(s)\}, \inf_{t-T\le s\le t}\{g(s)\}\}$$

In the following we set

$$\Gamma := \inf_{0\le s\le t-T}\{f(t-s)+g(s)\}$$

and

$$\Omega := \inf_{t-T\le s\le t}\{g(s)\}$$

So clearly

$$(f * g)(t) = \min\{\Gamma, \Omega\}$$

Since g is monotone increasing, we have $\Omega = g(t-T) = a(t-(T+A))$. The computation of Γ is subdivided further:

$$\Gamma = \min\{\gamma_1, \gamma_2\}$$

with

$$\gamma_1 := \inf_{0\le s\le A}\{f(t-s)+g(s)\}$$

and

$$\gamma_2 := \inf_{A\le s\le t-T}\{f(t-s)+g(s)\}$$

So we get

$$\begin{aligned}\gamma_1 &= \inf_{0\le s\le A}\{m\cdot(t-s-T)\}\\ &= m\cdot(t-(T+A))\end{aligned}$$

since $-s$ is monotone decreasing. Further we compute

$$\begin{aligned}\gamma_2 &= \inf_{A\le s\le t-T}\{m\cdot(t-s-T)+a(s-A)\}\\ &= \inf_{A\le s\le t-T}\{(a-m)\cdot s+m\cdot(t-T)-aA\}\end{aligned}$$

We have to consider two different cases:

$a \ge m$ Since $a - m \ge 0$, we have to choose s as small as possible, so we get

$$\gamma_2 = (a-m)\cdot A + m\cdot(t-T) - aA = m\cdot(t-(T+A))$$

$a \le m$ Since $a - m \le 0$, we have to choose s as large as possible, so we get

$$\gamma_2 = (a-m)\cdot(t-T) + m\cdot(t-T) - aA = a\cdot(t-(T+A))$$

Thus we have $\Gamma = \min a, m \cdot (t - (T + A))$. Recalling Ω, we have for $T + A \leq t \leq T + B$

$$(f * g)(t) = \min\{a, m\} \cdot (t - (T + A))$$

Step 3: $t - T \geq B$
We again have

$$(f * g)(t) = \min\{\inf_{0 \leq s \leq t-T}\{f(t-s) + g(s)\}, \inf_{t-T \leq s \leq t}\{g(s)\}\}$$

Again we set

$$\Gamma := \inf_{0 \leq s \leq t-T}\{f(t-s) + g(s)\}$$

and

$$\Omega := \inf_{t-T \leq s \leq t}\{g(s)\}$$

So clearly

$$(f * g)(t) = \min\{\Gamma, \Omega\}$$

Since g is monotone increasing, we have $\Omega = g(t-T) = b \cdot (t-(T+B)) + a \cdot (B-A)$. The computation of Γ is subdivided further:

$$\Gamma = \min\{\gamma_1, \gamma_2, \gamma_3\}$$

with

$$\gamma_1 := \inf_{0 \leq s \leq A}\{f(t-s) + g(s)\},$$

$$\gamma_2 := \inf_{A \leq s \leq B}\{f(t-s) + g(s)\}$$

and

$$\gamma_3 := \inf_{B \leq s \leq t-T}\{f(t-s) + g(s)\}$$

So we get

$$\begin{aligned} \gamma_1 &= \inf_{0 \leq s \leq A}\{m \cdot (t - s - T)\} \\ &= m \cdot (t - (T + A)) \end{aligned}$$

since $-s$ is monotone decreasing. Further we compute

$$\gamma_2 = \inf_{A \leq s \leq B}\{(a - m) \cdot s + m \cdot (t - T) - aA\}$$

We have to consider two different cases:

$a \geq m$ Since $a - m \geq 0$, we have to choose s as small as possible, so we get

$$\gamma_2 = (a - m) \cdot A + m \cdot (t - T) - aA = m \cdot (t - (T + A))$$

$a \leq m$ Since $a - m \leq 0$, we have to choose s as large as possible, so we get

$$\gamma_2 = (a-m) \cdot B + m \cdot (t-T) - aA = m \cdot (t - (T+B)) + a \cdot (B-A)$$

For γ_3 we obtain

$$\begin{aligned}
\gamma_3 &= \inf_{B \leq s \leq t-T} \{f(t-s) + g(s)\} \\
&= \inf_{B \leq s \leq t-T} \{m \cdot (t-s-T) + b \cdot (s-B) + a \cdot (B+A)\} \\
&= \inf_{B \leq s \leq t-T} \{(b-m) \cdot s + m \cdot (t-T) - bB + a(B-A)\}
\end{aligned}$$

We have to consider two different cases:

$b \geq m$ Since $b - m \geq 0$, we have to choose s as small as possible, so we get

$$\begin{aligned}
\gamma_3 &= (b-m) \cdot B + m \cdot (t-T) - bB + a(B-A) \\
&= m \cdot (t-(T+B)) + a \cdot (B-A)
\end{aligned}$$

$b \leq m$ Since $a - m \leq 0$, we have to choose s as large as possible, so we get

$$\begin{aligned}
\gamma_3 &= (b-m) \cdot (t-T) + m \cdot (t-T) - bB + a(B-A) \\
&= b \cdot (t-(T+B)) + a \cdot (B-A)
\end{aligned}$$

So

$$\gamma_3 = \min\{b, m\} \cdot (t - (T+B)) + a \cdot (B-A)$$

STEP II

Now we have to collect the results and decide, what the minimum is depending on the parameters.

First we consider $b \geq m$. Note that by our assumption this implies $a \geq m$. We have

$$\gamma_1 = \gamma_2 = m \cdot (t - (T+A))$$

and

$$\gamma_3 = m \cdot (t-(T+B)) + a \cdot (B-A)$$

Solving the inequality

$$\gamma_1 \leq \gamma_3$$

yields

$$\begin{aligned}
\gamma_1 &\leq \gamma_3 \\
m \cdot (t-(T+A)) &\leq m \cdot (t-(T+B)) + a \cdot (B-A) \\
m \cdot (B-A) &\leq a \cdot (B-A) \\
m &\leq a
\end{aligned}$$

since by assumption $B > A$. So we have

$$\Gamma = \gamma_1 = m \cdot (t - (T + A))$$

Recall, $(f * g)(t) = \min\{\Gamma, \Omega\}$, so we have to check the inequality $\Gamma \leq \Omega$. We have

$$\begin{aligned} m \cdot (t - (T + A)) &\leq b \cdot (t - (T + B)) + a \cdot (B - A) \\ (m - b) \cdot t &\leq -b \cdot (T + B) + m \cdot (T + A) + a(B - A) \\ (m - b) \cdot t &\leq (m - b) \cdot T + mA - bB + a(B - A) \\ t &\geq T + \frac{A \cdot (m - a) + B \cdot (b - a)}{m - b} \end{aligned}$$

where we have used $b > m$. We set

$$D := \frac{A \cdot (m - a) + B \cdot (b - a)}{m - b}$$

For $b = m$ we have

$$\begin{aligned} (m - b) \cdot t &\leq (m - b) \cdot T + mA - bB + a(B - A) \\ 0 &\leq (a - m) \cdot (B - A) \end{aligned}$$

and since $a \geq m$ (and $B > A$) this is always satisfied!
Now another question has to be answered, namely for which choices of the parameters do we have $D \leq B$? Recall, in this step we assume $t \geq T + B$.

$$\begin{aligned} \frac{A \cdot (m - a) + B \cdot (b - a)}{m - b} &\leq B \\ A \cdot (m - a) + B \cdot (b - a) &\geq B \cdot (m - b) \\ a \cdot (B - A) &\geq m \cdot (B - A) \\ a &\geq m \end{aligned}$$

So we have in this case $t + D \leq t + B$ and hence $(f * g)(t) = \min\{\Gamma, \Omega\} = \Gamma$.

Now we assume $b < m$ and still we have $t \geq T + B$. We have to discuss two different situations:

1. $b \leq a \leq m$
   In this case we have

$$\begin{aligned} \gamma_1 &= m \cdot (t - (T + A)) \\ \gamma_2 &= m \cdot (t - (T + B)) + a \cdot (B - A) \\ \gamma_3 &= b \cdot (t - (T + B)) + a \cdot (B - A) \end{aligned}$$

   Clearly, $\gamma_3 \leq \gamma_2$. So we check the relation of γ_1 and γ_3.

$$\begin{aligned} \gamma_3 &\leq \gamma_1 \\ b \cdot (t - (T + B)) + a \cdot (B - A) &\leq m \cdot (t - (T + A)) \\ 0 &\leq (m - b) \cdot t + b \cdot (T + B) - m \cdot (T + A) \\ 0 &\leq (m - b) \cdot (t - T) + bB - mA \end{aligned}$$

Since $B > A$, we can estimate

$$(m-b)\cdot(t-T)+bB-mA \geq (m-b)\cdot(t-T-A) \geq (m-b)\cdot(B-A) > 0$$

Hence $\Gamma = \gamma_3 = \Omega$.

2. $b \leq m \leq a$
   In this case we have

$$\begin{aligned}\gamma_1 &= m\cdot(t-(T+A)) \\ \gamma_2 &= m\cdot(t-(T+A)) \\ \gamma_3 &= b\cdot(t-(T+B))+a\cdot(B-A)\end{aligned}$$

   We note that $\Omega = \gamma_3$, so we only have to discuss when $\gamma_1 \leq \gamma_3$. But we did that already, we obtained

$$\begin{aligned}\gamma_1 &\leq \gamma_3 \\ t &\leq T+D\end{aligned}$$

   So we have

$$(f*g)(t) = \min\{\Gamma,\Omega\} = \Gamma = \begin{cases}\gamma_1 & t \leq T+D \\ \gamma_3 & t \geq T+D\end{cases}$$

Collecting all the information gathered during the proof yields the assumption.

□

Since we want to be able to predict the piecewise affine functions occuring in the computations, we collect them:

$$\begin{aligned}\eta_1 &= \min\{a,m\}\cdot(t-(T+A)) \\ \eta_2 &= \min\{b,m\}\cdot(t-(T+B))+a\cdot(B-A)\end{aligned}$$

Now we will try to write the min–plus convolution in one formula, expressed in terms of affine functions. We can express $(f*g)(t)$ as

$$(f*g)(t) = \min\{X,Y\}$$

with

$$X := \max\{0,\min\{a,m\}\cdot(t-(T+A))\} = \max\{0,\eta_1\}$$

and

$$Y := \max\{0,\min\{b,m\}\cdot(t-(T+B))+a\cdot(B-A)\} = \max\{0,\eta_2\}$$

It remains to be seen if one can extend these results to more general cases, e.g. more different slopes of the function g.

4.3 RLC and Nonconvex Piecewise Linear Function With 3 Slopes

We investigate the min–plus convolution of the parameterized families of functions f and g. Their graphs and the corresponding min–plus convolutions are depicted below. The functions' formal definition is:

$$f(x) = \begin{cases} 0 & x < T \\ m \cdot (x - T) & x \geq T \end{cases}$$

and

$$g(x) = \begin{cases} 0 & x < A \\ a \cdot (x - A) & A \leq x \leq B \\ b \cdot (x - B) + a \cdot (B - A) & B \leq x \leq D \\ c \cdot (x - D) + b \cdot (D - B) + a \cdot (B - A) & D \leq x \end{cases}$$

In the following we assume $a \geq b \geq c$ and $A < B < D$. As indicated above we are only interested in the case $A, T > 0$.
We want to compute for all $t \in \mathbb{R}$ the following quantity:

$$(f * g)(t) = \inf_{0 \leq s \leq t} \{f(t - s) + g(s)\}$$

that is the min–plus convolution of f and g. Since $f, g : \mathbb{R} \to \mathbb{R}_0^+$, we clearly have $f * g \geq 0$.
In the following we will show, that we get the following expressions for the min–plus convolution, assuming $T, A > 0$, $A < B$ and $a \geq b \geq c$:

1. Case $a \geq b \geq c \geq m$

$$(f * g)(t) = \begin{cases} 0 & t \leq T + A \\ m \cdot (t - (T + A)) & T + A \leq t \end{cases}$$

2. Case $a \geq b \geq m \geq c$

$$(f * g)(t) = \begin{cases} 0 & t \leq T + A \\ m \cdot (t - (T + A)) & T + A \leq t \leq T + O \\ c \cdot (t - (T + D)) & \\ \quad + b \cdot (D - B) + a \cdot (B - A) & T + O \leq t \end{cases}$$

with

$$O := \frac{D \cdot (b - c) + A \cdot (m - a) + B \cdot (a - b)}{m - c}$$

3. Case $a \geq m \geq b \geq c$
We define O as in Case 2. One can show $D \leq O$ iff

$$0 \leq b \cdot (D - B) + a \cdot (B - A) - m \cdot (D - A),$$

further set

$$M := \frac{B \cdot (b-a) + A(a-m)}{b-m}$$

and show $M \leq D$ iff

$$0 \geq b \cdot (D-B) + a \cdot (B-A) - m \cdot (D-A)$$

So we have either $M \leq D$ and thus $O \leq D$ or we have $M \geq D$ and $O \geq D$. So O and M always lie on the same side of D.
So let $M, O \leq D$, then we obtain

$$(f * g)(t) = \begin{cases} 0 & t \leq T+A \\ m \cdot (t-(T+A)) & T+A \leq t \leq T+M \\ b \cdot (t-(T+B)) + a \cdot (B-A) & T+M \leq t \leq T+D \\ c \cdot (t-(T+D)) & \\ \quad +b \cdot (D-B) + a \cdot (B-A) & T+D \leq t \end{cases}$$

Now let $M, O \geq D$; this yields

$$(f * g)(t) = \begin{cases} 0 & t \leq T+A \\ m \cdot (t-(T+A)) & T+A \leq t \leq T+O \\ c \cdot (t-(T+D)) & \\ \quad +b \cdot (D-B) + a \cdot (B-A) & T+O \leq t \end{cases}$$

4. Case $m \geq a \geq b \geq c$

$$(f * g)(t) = \begin{cases} 0 & t \leq T+A \\ a \cdot (t-(T+A)) & T+A \leq t \leq T+B \\ b \cdot (t-(T+B)) + a \cdot (B-A) & T+B \leq t \leq T+D \\ c \cdot (t-(T+D)) & \\ \quad +b \cdot (D-B) + a \cdot (B-A) & T+D \leq t \end{cases}$$

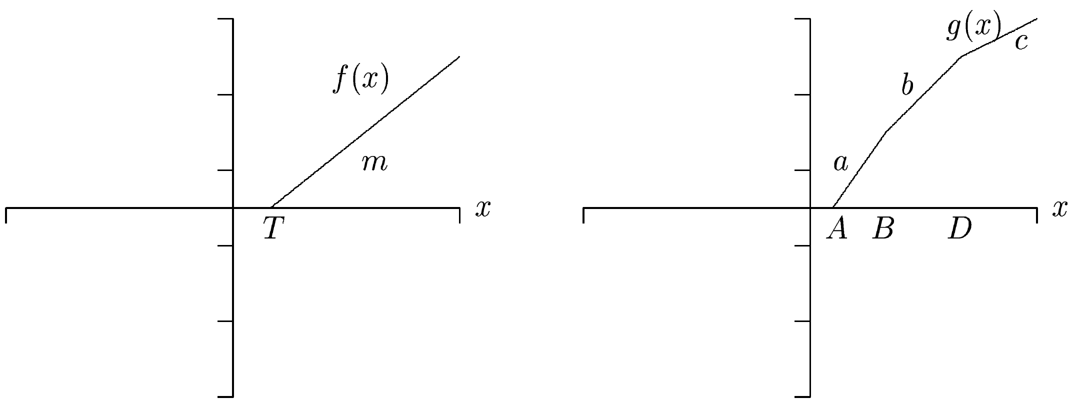

Figure 4.7: RLC and Nonconvex Piecewise Linear Function With 3 Slopes

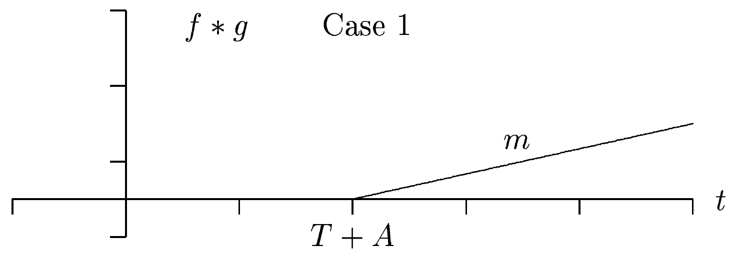

Figure 4.8: Min–plus Convolution of RLC and NPL Function Case 1

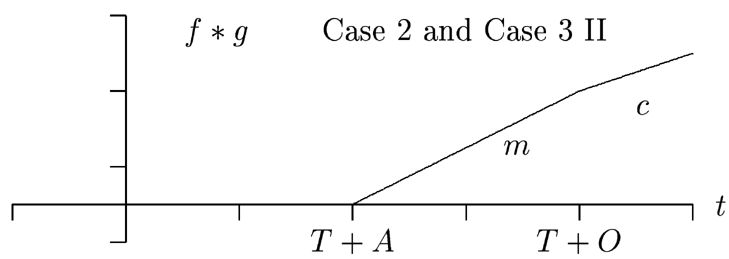

Figure 4.9: Min–plus Convolution of RLC and NPL Function Case 2 and 3 II

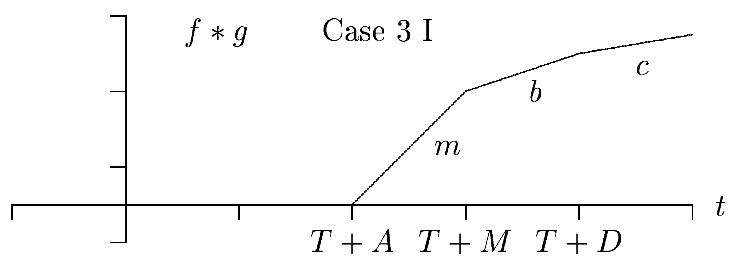

Figure 4.10: Min–plus Convolution of RLC and NPL Function Case 3 I

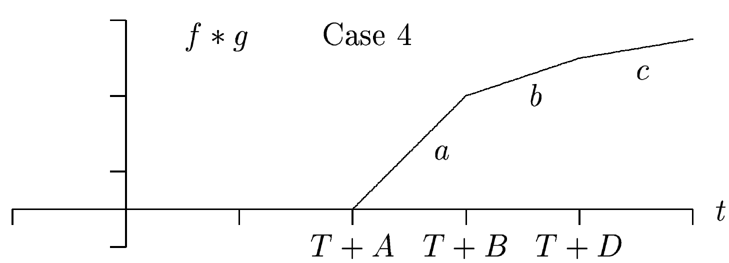

Figure 4.11: Min–plus Convolution of RLC and NPL Function Case 4

Proof We wish to compute

$$(f * g)(t) = \inf_{0 \leq s \leq t} \{f(t-s) + g(s)\}$$

As already mentioned, we have $f * g \geq 0$ in our case. We will compute $f * g$ in steps. We note that $t - s < T$ iff $s > t - T$.

STEP I

Step 1: $t - T \leq A$

We have

$$\begin{aligned}
(f * g)(t) &= \inf_{0 \leq s \leq t} \{f(t-s) + g(s)\} \\
&= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, \inf_{t-T \leq s \leq t} \{f(t-s) + g(s)\}\} \\
&= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, \inf_{t-T \leq s \leq t} \{g(s)\}\} \\
&= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, 0\} \\
&= 0
\end{aligned}$$

Step 2: $A \leq t - T \leq B$

$$\begin{aligned}
(f * g)(t) &= \inf_{0 \leq s \leq t} \{f(t-s) + g(s)\} \\
&= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, \inf_{t-T \leq s \leq t} \{f(t-s) + g(s)\}\} \\
&= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, \inf_{t-T \leq s \leq t} \{g(s)\}\} \\
&= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, g(t-T)\} \\
&= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, a \cdot (t - (T + A))\}
\end{aligned}$$

We set

$$\Gamma := \inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}$$

This can be further subdivided, if we set

$$\begin{aligned}
\gamma_1 &= \inf_{0 \leq s \leq A} \{f(t-s) + g(s)\} \\
\gamma_2 &= \inf_{A \leq s \leq t-T} \{f(t-s) + g(s)\}
\end{aligned}$$

Obviously it holds $\Gamma = \min\{\gamma_1, \gamma_2\}$.
γ_1 and γ_2 can be computed easily, since on the corresponding sets we know (by construction) which part of f and g is valid.

$$\gamma_1 = \inf_{0 \leq s \leq A} \{m \cdot (t - s - T) + 0\} = m \cdot (t - (A + T))$$

For γ_2 we have to distinguish two cases:

$a \geq m$

$$\gamma_2 = \inf_{A \leq s \leq t-T} \{m \cdot (t-s-T) + a(s-A)\} = m \cdot (t-(A+T))$$

$a \leq m$

$$\gamma_2 = \inf_{A \leq s \leq t-T} \{m \cdot (t-s-T) + a(s-A)\} = a \cdot (t-(A+T))$$

So we have

$$\Gamma = \min\{\gamma_1, \gamma_2\} = \min\{a, m\} \cdot (t-(T+A))$$

Recall, $(f * g)(t) = \min\{\Gamma, a \cdot (t-(T+A))\}$, so we get in this case

$$(f * g)(t) = \min\{a, m\} \cdot (t-(T+A))$$

Step 3: $B \leq t - T \leq D$

Again, we have by the monotonicity of g

$$\begin{aligned} (f * g)(t) &= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, g(t-T)\} \\ &= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, b \cdot (t-(T+B)) + a \cdot (B-A)\} \end{aligned}$$

Again we set

$$\Gamma := \inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}$$

This can be further subdivided, setting

$$\begin{aligned} \gamma_1 &= \inf_{0 \leq s \leq A} \{f(t-s) + g(s)\} \\ \gamma_2 &= \inf_{A \leq s \leq B} \{f(t-s) + g(s)\} \\ \gamma_3 &= \inf_{B \leq s \leq t-T} \{f(t-s) + g(s)\} \end{aligned}$$

Obviously it holds $\Gamma = \min\{\gamma_1, \gamma_2, \gamma_3\}$.
γ_1, γ_2 and γ_3 can be computed easily, since on the corresponding sets we know (by construction) which part of f and g is valid.
We have again

$$\gamma_1 = \inf_{0 \leq s \leq A} \{m \cdot (t-s-T) + 0\} = m \cdot (t-(A+T))$$

For γ_2 and γ_3 we have to distinguish 2 cases, which is not surprising.
We investigate γ_3 in a similar manner.

$$\begin{aligned} \gamma_2 &= \inf_{A \leq s \leq B} \{f(t-s) + g(s)\} \\ &= \inf_{A \leq s \leq B} \{m(t-s-T) + a(s-A)\} \end{aligned}$$

$a \geq m$

$$\gamma_2 = m \cdot (t-(T+A))$$

$a < m$

$$\gamma_2 = m \cdot (t - (T + B)) + a \cdot (B - A)$$

We now can rewrite γ_2 in order to get a formula, that covers both cases:

$$m \cdot (t - (T + A)) = m \cdot (t - (T + B)) + m(B - A)$$

So we can write

$$\gamma_2 = m \cdot (t - (T + B)) + \min\{m, a\} \cdot (B - A)$$

We investigate γ_3 in a similar manner.

$$\begin{aligned} \gamma_3 &= \inf_{B \le s \le t-T} \{f(t - s) + g(s)\} \\ &= \inf_{B \le s \le t-T} \{m(t - s - T) + b(s - B) + a(B - A)\} \end{aligned}$$

$b \geq m$

$$\gamma_3 = m \cdot (t - (T + B)) + a \cdot (B - A)$$

$b < m$

$$\gamma_3 = b \cdot (t - (T + B)) + a \cdot (B - A)$$

So we have

$$\gamma_3 = \min\{b, m\}(t - (T + B)) + a \cdot (B - A)$$

In order to compute $\Gamma = \min\{\gamma_1, \gamma_2, \gamma_3\}$ it is important, how the parameters relate to each other! Recall that we always assume $a \geq b$. We consider three different cases:

a) $a \geq b \geq m$ We have

$$\begin{aligned} \gamma_1 &= m \cdot (t - (A + T)) \\ \gamma_2 &= m \cdot (t - (A + T)) \\ \gamma_3 &= m \cdot (t - (T + B)) + a \cdot (B - A) \end{aligned}$$

So we have to decide, for which t we have $\gamma_1 = \gamma_2 \leq \gamma_3$.

$$\begin{aligned} m \cdot (t - (A + T)) &\leq m \cdot (t - (T + B)) + a \cdot (B - A) \\ 0 &\leq m \cdot (A - B) + a \cdot (B - A) \\ 0 &\leq (B - A) \cdot (a - m) \end{aligned}$$

Since $B > A$ by assumption, we have $\Gamma = \gamma_1$ in this case!
Recall,

$$(f * g)(t) = \min\{\Gamma, b \cdot (t - (T + B)) + a \cdot (B - A)\}$$

So we must decide for which t we have

$$\begin{aligned} m\cdot(t-(A+T)) &\leq b\cdot(t-(T+B))+a\cdot(B-A) \\ (m-b)\cdot t &\leq T\cdot(m-b)+B(a-b)+A(m-a) \\ t &\geq T+\frac{B(a-b)+A(m-a)}{m-b} \end{aligned}$$

where we have made use of $m-b<0$. For $m=b$ we have

$$0 \leq B(a-b)+A(b-a)$$

which is also true in this case. We set

$$M := \frac{B(b-a)+A(a-m)}{b-m}$$

We are still in the case $B \leq t-T \leq D$, so we have $B+T \leq T$ and $t \leq T+D$. So we have to decide, whether $M \leq B$ and if not, whether $M \leq D$ for certain parameters. We have in particular $b>m$

$$\begin{aligned} \frac{B(b-a)+A(a-m)}{b-m} &\leq B \\ B(b-a)+A(a-m) &\leq B(b-m) \\ B(m-a)+A(a-m) &\leq 0 \\ (A-B)(a-m) &\leq 0 \end{aligned}$$

which is true for this parameter set. We require $a \geq m$ for this inequality to hold!
We have $M \leq B$, thus $T+M \leq T+B \leq t$ and we get

$$(f*g)(t)=\Gamma=m\cdot(t-(A+T))$$

b) $a \geq m > b$ We have

$$\begin{aligned} \gamma_1 &= m\cdot(t-(A+T)) \\ \gamma_2 &= m\cdot(t-(A+T)) \\ \gamma_3 &= b\cdot(t-(T+B))+a\cdot(B-A) \end{aligned}$$

So we have to decide, for which t we have $\gamma_1=\gamma_2 \leq \gamma_3$. But now we have $b<m$, hence $m-b>0$.

$$\begin{aligned} m\cdot(t-(A+T)) &\leq b\cdot(t-(T+B))+a\cdot(B-A) \\ (m-b)\cdot t &\leq T\cdot(m-b)+B(a-b)+A(m-a) \\ t &\leq T+\frac{B(a-b)+A(m-a)}{m-b} \end{aligned}$$

For $m = b$ we still have

$$0 \leq B(a-b) + A(b-a)$$

which is also true in this case. We set

$$M := \frac{B(b-a) + A(a-m)}{b-m}$$

We have to decide, whether $M \leq B$ and if not, whether $M \leq D$ for certain parameters. We have in particular $b < m$

$$\begin{aligned} \frac{B(b-a) + A(a-m)}{b-m} &\leq B \\ B(b-a) + A(a-m) &\geq B(b-m) \\ B(m-a) + A(a-m) &\geq 0 \\ (A-B)(a-m) &\geq 0 \end{aligned}$$

which is false for this parameter set. So we have $M > B$. So we have to check $M \leq D$:

$$\begin{aligned} \frac{B(b-a) + A(a-m)}{b-m} &\leq D \\ B(b-a) + A(a-m) &\geq D(b-m) \\ b(B-D) + a(A-B) + m(D-A) &\geq 0 \end{aligned}$$

It is not immediately clear if this relation holds for all parameter sets. Thus it should be checked individually.
Since $\gamma_3 = g(t-T)$ in this case, we have for $B < M < D$

$$(f*g)(t) = \begin{cases} m \cdot (t - (A+T)) & T+B \leq t \leq T+M \\ b \cdot (t - (T+B)) + a \cdot (B-A) & T+M \leq t \leq T+D \end{cases}$$

If instead $B < D < M$ we simply obtain

$$(f*g)(t) = m \cdot (t - (A+T)) \qquad T+B \leq t \leq T+D$$

c) $m > a \geq b$ We have

$$\begin{aligned} \gamma_1 &= m \cdot (t - (T+A)) \\ \gamma_2 &= m \cdot (t - (T+B)) + a \cdot (B-A) \\ \gamma_3 &= b \cdot (t - (T+B)) + a \cdot (B-A) \end{aligned}$$

Clearly we have $\gamma_2 > \gamma_3$. So we have to decide for which t we have $\gamma_1 \leq \gamma_3$.

$$\begin{aligned} m \cdot (t - (T+A)) &\leq b \cdot (t - (T+B)) + a \cdot (B-A) \\ (m-b) \cdot t &\leq T \cdot (m-b) + B(a-b) + A(m-a) \\ t &\leq T + \frac{B(a-b) + A(m-a)}{m-b} \end{aligned}$$

Again we set

$$M := \frac{B(b-a) + A(a-m)}{b-m}$$

We have to decide, whether $M \leq B$ and if not, whether $M \leq D$ for certain parameters. We have in particular $b < m$

$$\begin{aligned} \frac{B(b-a) + A(a-m)}{b-m} &\leq B \\ B(b-a) + A(a-m) &\geq B(b-m) \\ B(m-a) + A(a-m) &\geq 0 \\ (A-B)(a-m) &\geq 0 \end{aligned}$$

which is true in this case. So we have $M \leq B$.
So we have

$$(f{*}g)(t) = \; b \cdot (t - (T+B)) + a \cdot (B-A) \qquad T+B \;\leq\; t \;\leq\; T+D$$

Step 4: $D \leq t - T$

Again, we have by the monotonicity of g

$$\begin{aligned} (f * g)(t) &= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, g(t-T)\} \\ &= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, \\ &\qquad c \cdot (t - (T+D)) + b(D-B) + a \cdot (B-A)\} \end{aligned}$$

Again we set

$$\Gamma := \inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}$$

This can be further subdivided, setting

$$\begin{aligned} \gamma_1 &= \inf_{0 \leq s \leq A} \{m \cdot (t-s-T) + g(s)\} \\ \gamma_2 &= \inf_{A \leq s \leq B} \{m \cdot (t-s-T) + g(s)\} \\ \gamma_3 &= \inf_{B \leq s \leq D} \{m \cdot (t-s-T) + g(s)\} \\ \gamma_4 &= \inf_{D \leq s \leq t-T} \{m \cdot (t-s-T) + g(s)\} \end{aligned}$$

Obviously it holds $\Gamma = \min\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$.
γ_1, γ_2, γ_3 and γ_4 can be computed easily, since on the corresponding sets we know (by construction) which part of f and g is valid.
We have

$$\gamma_1 = \inf_{0 \leq s \leq A} \{m \cdot (t-s-T) + 0\} = m \cdot (t - (A+T))$$

and

$$\gamma_2 = m \cdot (t - (T+B)) + \min\{m, a\} \cdot (B-A)$$

as seen in *Step* 3. For γ_3 and γ_4 we have to distinguish 2 cases, which is not surprising.

We investigate γ_3

$$\gamma_3 = \inf_{B \le s \le D} \{m \cdot (t - s - T) + b(s - B) + a \cdot (B - A)\}$$

$b \ge m$

$$\gamma_3 = m \cdot (t - (T + B)) + a \cdot (B - A)$$

$b < m$

$$\gamma_3 = m \cdot (t - (T + D)) + b \cdot (D - B) + a \cdot (B - A)$$

We now can rewrite γ_3 in order to get a formula, that covers both cases:

$$m \cdot (t - (T + B)) = m \cdot (t - (T + D)) + m(D - B)$$

So we can write

$$\gamma_3 = m \cdot (t - (T + D)) + \min\{m, b\} \cdot (D - B) + a \cdot (B - A)$$

We investigate γ_4 in a similar manner.

$$\gamma_4 = \inf_{D \le s \le t-T} \{m \cdot (t - s - T) + c(s - D) + b \cdot (D - B) + a \cdot (B - A)\}$$

$c \ge m$

$$\gamma_4 = m \cdot (t - (T + B)) + b \cdot (D - B) + a \cdot (B - A)$$

$c < m$

$$\gamma_4 = c \cdot (t - (T + D)) + b \cdot (D - B) + a \cdot (B - A)$$

So we have

$$\gamma_4 = \min\{c, m\} \cdot (t - (T + D)) + b \cdot (D - B) + a \cdot (B - A)$$

In order to compute $\Gamma = \min\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ it is important, how the parameters relate to each other! Recall that we always assume $a \ge b \ge c$. We consider four different cases:

a) $a \ge b \ge c \ge m$
   We have

$$\begin{aligned} \gamma_1 &= m \cdot (t - (T + A)) \\ \gamma_2 &= m \cdot (t - (T + A)) \\ \gamma_3 &= m \cdot (t - (T + D)) + \min\{m, b\} \cdot (D - B) + a \cdot (B - A) \\ \gamma_4 &= m \cdot (t - (T + D)) + b \cdot (D - B) + a \cdot (B - A) \end{aligned}$$

Obviously we have $\gamma_3 \le \gamma_4$. So we have to decide when $\gamma_1 \le \gamma_3$.

$$\begin{aligned} m \cdot (t - (T + A)) &\le m \cdot (t - (T + B)) + a \cdot (B - A) \\ 0 &\le (B - A) \cdot (a - m) \end{aligned}$$

This holds true in this case. So we have $\Gamma = \gamma_1$. Further we have $g(t-T) = c\cdot(t-(T+D)) + b(D-B) + a\cdot(B-A) \geq \gamma_3$. So

$$(f * g)(t) = \min\{\Gamma, g(t-T)\} = m\cdot(t-(T+A))$$

b) $a \geq b \geq m > c$
We have

$$\begin{aligned}
\gamma_1 &= m\cdot(t-(T+A)) \\
\gamma_2 &= m\cdot(t-(T+A)) \\
\gamma_3 &= m\cdot(t-(T+D)) + \min\{m,b\}\cdot(D-B) + a\cdot(B-A) \\
\gamma_4 &= c\cdot(t-(T+D)) + b\cdot(D-B) + a\cdot(B-A)
\end{aligned}$$

Like in a) one shows $\gamma_1 \leq \gamma_3$.
We investigate when $\gamma_3 \leq \gamma_4$.

$$\begin{aligned}
\begin{matrix} m\cdot(t-(T+D)) \\ +m\cdot(D-B) + a\cdot(B-A) \end{matrix} &\leq \begin{matrix} c\cdot(t-(T+D)) \\ +b\cdot(D-B) + a\cdot(B-A) \end{matrix} \\
(m-c)t &\leq (T+D)(m-c) + (b-m)(D-B) \\
t &\leq T+D+\frac{(b-m)(D-B)}{m-c}
\end{aligned}$$

We set

$$N := \frac{(b-m)(D-B)}{m-c} \geq 0$$

We have to decide when $\gamma_1 \leq \gamma_4$.

$$\begin{aligned}
m\cdot(t-(T+A)) &\leq c\cdot(t-(T+D)) + b\cdot(D-B) + a\cdot(B-A) \\
(m-c)\cdot t &\leq T(m-c) + D(b-c) + B(a-b) + A(m-a) \\
t &\leq T + \frac{D(b-c)+B(a-b)+A(m-a)}{m-c}
\end{aligned}$$

where we have used $m > c$. We set

$$O := \frac{D(b-c)+B(a-b)+A(m-a)}{m-c}$$

We have to decide when $D \leq O$, since then $T+D \leq t \leq T+O$ is another valid subcase.

$$\begin{aligned}
D &\leq \frac{D(b-c)+B(a-b)+A(m-a)}{m-c} \\
D(m-c) &\leq D(b-c)+B(a-b)+A(m-a) \\
0 &\leq a(B-A)+b(D-B)+m(A-D) \\
0 &\geq b(B-D)+a(B-A)+m(D-A) \qquad (4.1)
\end{aligned}$$

Note, that we have encountered this condition before, in *Step* 3 b), with *different* inequality sign. We wanted to decide, whether $M \leq D$ and this condition was sufficient to guarantee this. So we have either $M \leq D$ and then $O \leq D$ or $M \geq D$ and then $D \leq O$.
One can show $D + N \leq O$ in this case, since $m > c$; in particular this implies $D \leq O$.

$$\begin{aligned}
D + \frac{(b-m)(D-B)}{m-c} &\leq \frac{D(b-c) + B(a-b) + A(m-a)}{m-c} \\
D(m-c) + (b-m)(D-B) &\leq D(b-c) + B(a-b) + A(m-a) \\
0 &\leq B(a-m) + A(m-a) \\
0 &\leq (a-m)(B-A)
\end{aligned}$$

Since $N \geq 0$, this means that one should be able to estimate (4.1) in this case, showing it to hold.
Since $\gamma_1 \leq \gamma_3$, we have for $T + D \leq T + D + N \leq T + O$

$$(f{*}g)(t) = \begin{cases} m \cdot (t - (T + A)) & T + D \leq t \leq T + D + N \\ m \cdot (t - (T + A)) & T + D + N \leq t \leq T + O \\ c \cdot (t - (T + D)) \quad cm & \\ \quad + b \cdot (D - B) + a \cdot (B - A) & T + O \leq t \end{cases}$$

c) $a \geq m > b \geq c$
We have

$$\begin{aligned}
\gamma_1 &= m \cdot (t - (T + A)) \\
\gamma_2 &= m \cdot (t - (T + A)) \\
\gamma_3 &= m \cdot (t - (T + D)) + b \cdot (D - B) + a \cdot (B - A) \\
\gamma_4 &= c \cdot (t - (T + D)) + b \cdot (D - B) + a \cdot (B - A)
\end{aligned}$$

Obviously we have $\gamma_4 \leq \gamma_3$. As in b) we have

$$\begin{aligned}
\gamma_1 &\leq \gamma_3 \\
m \cdot (t - (T + A)) &\leq c \cdot (t - (T + D)) + b \cdot (D - B) + a \cdot (B - A) \\
(m - c) \cdot t &\leq T(m-c) + D(b-c) + B(a-b) + A(m-a) \\
t &\leq T + O
\end{aligned}$$

We set

$$O := \frac{D(b-c) + B(a-b) + A(m-a)}{m-c}$$

As in b we have to decide $D \leq O$, so we have to check for each parameter set the inequality

$$b(B - D) + a(B - A) + m(D - A) \leq 0$$

Recall, that we defined M and that either $M, O \leq D$ or $M, O \geq D$. We can write

$$(f*g)(t) = \begin{cases} m \cdot (t-(T+A)) & T+D \leq t < T+O \\ c \cdot (t-(T+D)) & \\ +b \cdot (D-B) + a \cdot (B-A) & T+O \leq t \end{cases}$$

If $O < D$, then $\{t \mid T+D \leq t \leq T+O\} = \emptyset$, so we just have

$$(f*g)(t) = \; c \cdot (t-(T+D)) + b \cdot (D-B) + a \cdot (B-A) \qquad T+D \leq t$$

d) $m > a \geq b \geq c$
We have

$$\begin{aligned} \gamma_1 &= m \cdot (t-(T+A)) \\ \gamma_2 &= m \cdot (t-(T+B)) + a \cdot (B-A) \\ \gamma_3 &= m \cdot (t-(T+D)) + b \cdot (D-B) + a \cdot (B-A) \\ \gamma_4 &= c \cdot (t-(T+D)) + b \cdot (D-B) + a \cdot (B-A) \end{aligned}$$

We can rewrite γ_2:

$$\begin{aligned} \gamma_2 &= m \cdot (t-(T+B)) + a \cdot (B-A) \\ &= m \cdot (t-(T+A)) + (a-m) \cdot (B-A) \end{aligned}$$

Clearly, since $a < m$, we have $\gamma_2 < \gamma_1$; obviously, $\gamma_4 \leq \gamma_3$. So we have to discuss $\gamma_2 \leq \gamma_4$.

$$\begin{aligned} m \cdot (t-(T+B)) + a \cdot (B-A) &\leq \begin{array}{c} c \cdot (t-(T+D)) \\ +b \cdot (D-B) + a \cdot (B-A) \end{array} \\ (m-c) \cdot t &\leq T(m-c) - cD + mB + b \cdot (D-B) \\ t &\leq T + \frac{B(m-b) + D(b-c)}{m-c} \end{aligned}$$

We set

$$P := \frac{B(m-b) + D(b-c)}{m-c}$$

We have to decide, if $P \leq D$.

$$\begin{aligned} \frac{B(m-b) + D(b-c)}{m-c} &\leq D \\ B(m-b) + D(b-c) &\leq D(m-c) \\ (B-D)(m-b) &\leq 0 \end{aligned}$$

This is satisfied, since $m > b$ and $B \geq D$. We conclude

$$(f*g)(t) = \; c \cdot (t-(T+D)) + b \cdot (D-B) + a \cdot (B-A) \qquad T+D \leq t$$

We obtain the claim by collecting all the results from the corresponding cases in each step. □

Now we can again collect all the piecewise linear functions appearing in the proof.

$$\begin{aligned}
\eta_1(t) &= m \cdot (t - (T + A)) \\
\eta_2(t) &= \min\{a, m\} \cdot (t - (T + A)) \\
\eta_3(t) &= m \cdot (t - (T + B)) + \min\{a, m\} \cdot (B - A) \\
\eta_4(t) &= \min\{b, m\}(t - (T + B)) + a \cdot (B - A) \\
\eta_5(t) &= m \cdot (t - (T + D)) + \min\{b, m\} \cdot (D - B) + a \cdot (B - A) \\
\eta_5(t) &= \min\{c, m\} \cdot (t - (T + D)) + b \cdot (D - B) + a \cdot (B - A)
\end{aligned}$$

We have

$$(f * g)(t) = \min\{\eta_1(t)^+, \eta_2(t)^+, \eta_3(t)^+, \eta_4(t)^+, \eta_5(t)^+, \eta_6(t)^+\}$$

where $h(t)^+ = \max\{0, h(t)\}$. There seems to be a pattern that allows us to construct these piecewise linear functions from the original functions f and g.

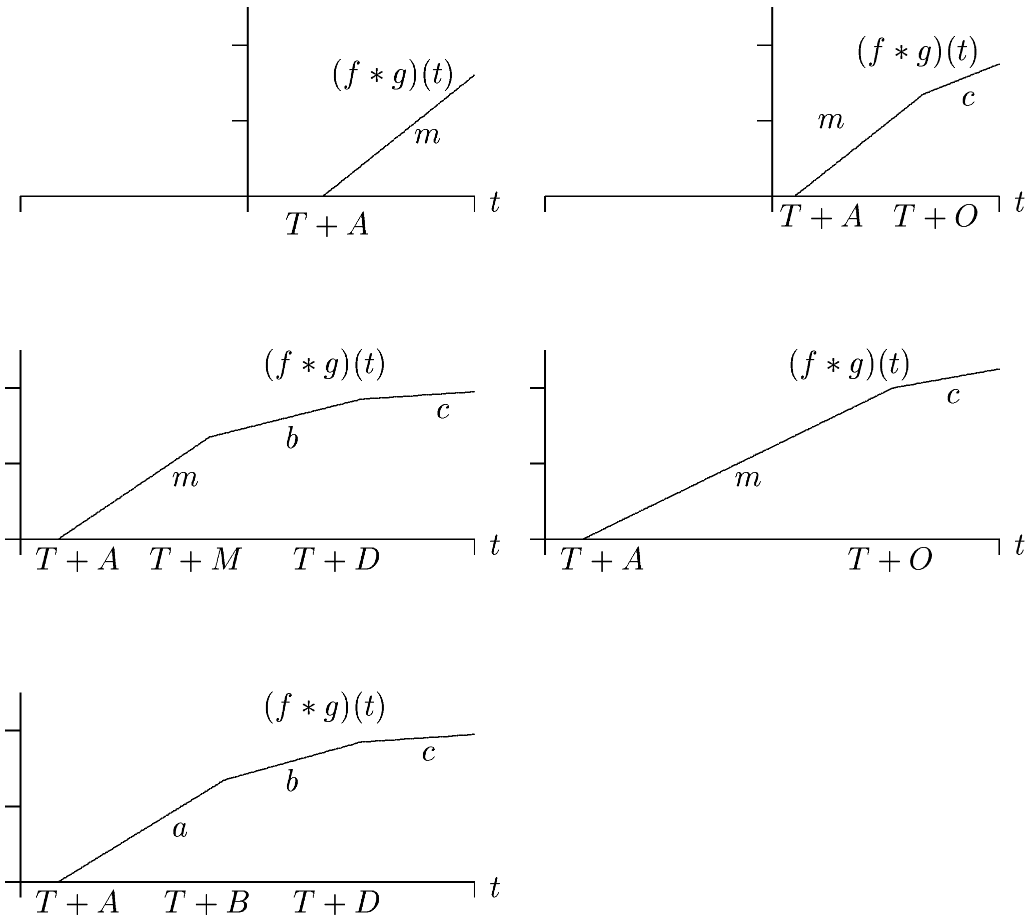

Figure 4.12: Graphs of $(f*g)$ in the Various Cases Scaled Better

4.4 Two Nonconvex Piecewise Linear Functions With 2 Slopes

We investigate the min–plus convolution of the parameterized families of functions f and g. Their graphs are depicted below.

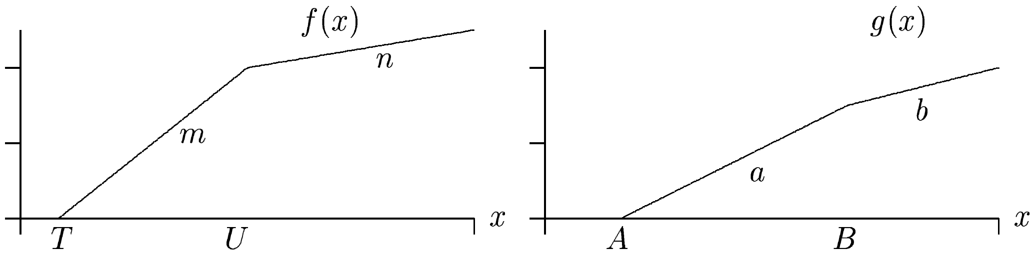

Figure 4.13: Two Nonconvex Piecewise Linear Functions

The functions' formal definition is:

$$f(x) = \begin{cases} 0 & x < T \\ m \cdot (x - T) & T \leq x \leq U \\ n \cdot (x - U) + m(U - T) & U < x \end{cases}$$

and

$$g(x) = \begin{cases} 0 & x < A \\ a \cdot (x - A) & A \leq x \leq B \\ b \cdot (x - A) + a(B - A) & B < x \end{cases}$$

In the following we assume $m \geq n$ and $a \geq b$, as well as $0 \leq T < U$ and $0 \leq A < B$. We want to compute for all $t \in \mathbb{R}$ the following quantity:

$$(f * g)(t) = \inf_{0 \leq s \leq t} \{f(t - s) + g(s)\}$$

that is the min–plus convolution of f and g. It turns out that for the min–plus convolution of these two functions the relation of $B - A$ and $U - T$ will be very important for some parameter sets, so we distinguish two subcases for each parameter set.
Without loss of generality we may assume $a \geq m$, since if $a < m$ we simply interchange the roles of g and f, renaming the corresponding parameters!
We will derive the following formulas for the min–plus convolution:

$$\boxed{a \geq b \geq m \geq n \qquad B - A \geq U - T}$$

$$(f * g)(t) = \begin{cases} 0 & t \leq T + A \\ m(t - (T + A)) & T + A < t \leq U + A \\ n \cdot (t - (A + U)) + m(U - T) & U + A < t \end{cases}$$

$$\boxed{a \geq b \geq m \geq n \qquad B - A < U - T}$$

$$(f*g)(t) = \begin{cases} 0 & t \leq T+A \\ m(t-(T+A)) & T+A < t \leq U+A \\ n \cdot (t-(A+U)) + m(U-T) & U+A < t \end{cases}$$

$$\boxed{a \geq m > b \geq n \qquad B - A \geq U - T}$$

$$(f*g)(t) = \begin{cases} 0 & t \leq T+A \\ m(t-(T+A)) & T+A < t \leq U+A \\ n \cdot (t-(A+U)) + m(U-T) & U+A < t \end{cases}$$

$$\boxed{\begin{array}{c} a \geq m > b \geq n \qquad B - A < U - T \\ (a-b)(B-A) + (b-m)(U-T) \leq 0 \end{array}}$$

We set

$$\begin{aligned} S &:= \frac{B(a-b) + A(m-a)}{m-b} \\ V &:= \frac{T(m-b) + U(n-m) + B(a-b) + A(n-a)}{n-b} \end{aligned}$$

$$(f*g)(t) = \begin{cases} 0 & t \leq T+A \\ m \cdot (t-(T+A)) & T+A \leq t \leq T+S \\ b \cdot (t-(T+B)) + a \cdot (B-A) & T+S < t \leq V \\ n \cdot (t-(A+U)) + m(U-T) & V < t \end{cases}$$

$$\boxed{\begin{array}{c} a \geq m > b \geq n \qquad B - A < U - T \\ (a-b)(B-A) + (b-m)(U-T) > 0 \end{array}}$$

$$(f*g)(t) = \begin{cases} 0 & t \leq T+A \\ m \cdot (t-(T+A)) & T+A \leq t \leq A+U \\ n \cdot (t-(A+U)) + m(U-T) & A+U < t \end{cases}$$

$$\boxed{a \geq m \geq n > b \qquad B - A \geq U - T}$$

We set

$$M := \frac{T(m-b) + B(a-b) + A(n-a) + U(n-m)}{n-b}$$

$$(f*g)(t) = \begin{cases} 0 & t \leq T+A \\ m \cdot (t - (T+A)) & T+A \leq t \leq U+A \\ n \cdot (t - (A+U)) + m(U-T) & U+A < t \leq M \\ b \cdot (t - (T+B)) + a(B-A) & M < t \end{cases}$$

$$\boxed{\begin{array}{l} a \geq m \geq n > b \qquad B - A < U - T \\ (a-b)(B-A) + (b-m)(U-T) \leq 0 \end{array}}$$

$$(f*g)(t) = \begin{cases} 0 & t \leq T+A \\ m \cdot (t - (T+A)) & T+A \leq t \leq T+S \\ b \cdot (t - (T+B)) + a \cdot (B-A) & T+S < t \end{cases}$$

$$\boxed{\begin{array}{l} a \geq m \geq n > b \qquad B - A < U - T \\ (a-b)(B-A) + (b-m)(U-T) > 0 \end{array}}$$

We set

$$\begin{aligned} S &:= \frac{B(a-b) + A(m-a)}{m-b} \\ V &:= \frac{T(m-b) + U(n-m) + B(a-b) + A(n-a)}{n-b} \end{aligned}$$

$$(f*g)(t) = \begin{cases} 0 & t \leq T+A \\ m \cdot (t - (T+A)) & T+A \leq t \leq A+U \\ n \cdot (t - (A+U)) + m(U-T) & A+U < t \leq V \\ b \cdot (t - (T+B)) + a \cdot (B-A) & V < t \end{cases}$$

Proof

STEP I

We wish to compute

$$(f * g)(t) = \inf_{0 \leq s \leq t} \{f(t-s) + g(s)\}$$

As already mentioned, we have $f * g \geq 0$ in our case. We will compute $f * g$ in steps. We note that $t - s < T$ iff $s > t - T$.

Step 1: $t - T \leq A$

We have

$$\begin{aligned}
(f * g)(t) &= \inf_{0 \leq s \leq t} \{f(t-s) + g(s)\} \\
&= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, \inf_{t-T \leq s \leq t} \{f(t-s) + g(s)\}\} \\
&= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, \inf_{t-T \leq s \leq t} \{g(s)\}\} \\
&= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, 0\} \\
&= 0
\end{aligned}$$

Step 2: $A \leq t - T \leq B$

$$\begin{aligned}
(f * g)(t) &= \inf_{0 \leq s \leq t} \{f(t-s) + g(s)\} \\
&= \min\{\inf_{0 \leq s \leq t-T} \{f(t-s) + g(s)\}, \inf_{t-T \leq s \leq t} \{f(t-s) + g(s)\}\} \\
&= \min\{\inf_{0 \leq s \leq A} \{f(t-s) + g(s)\}, \inf_{A \leq s \leq t-T} \{f(t-s) + g(s)\}, \inf_{t-T \leq s \leq t} \{g(s)\}\} \\
&= \min\{f(t-A), \inf_{A \leq s \leq t-T} \{f(t-s) + g(s)\}, g(t-T)\} \\
&= \min\{f(t-A), \inf_{A \leq s \leq t-T} \{f(t-s) + g(s)\}, a \cdot (t - (T+A))\}
\end{aligned}$$

where we have used the monotonicity of both f and g.

We have

$$f(t-A) = \begin{cases} m \cdot (t - (T+A)) & T + A \leq t \leq U + A \\ n \cdot (t - (A+U)) + m \cdot (U - T) & U + A \leq t \end{cases}$$

We set

$$\Gamma := \inf_{A \leq s \leq t-T} \{f(t-s) + g(s)\}$$

Then, clearly

$$(f * g)(t) = \min\{f(t-A), a \cdot (t - (T+A)), \Gamma\}$$

In order to compute this quantity we have to decide, if $t-s=U$ is possible, since at this point f changes its behavior. $t-s=U$ is equivalent to $s=t-U$. So we set

$$\begin{aligned}\gamma_1 &= \inf_{A\leq s\leq t-U}\{f(t-s)+g(s)\}\\ \gamma_2 &= \inf_{t-U\leq s\leq t-T}\{f(t-s)+g(s)\}\end{aligned}$$

For this to make sense, we require $A\leq t-U$, i.e. $A+U\leq t$. Since $T<U$ we have $A+T<A+U$. But the other restriction on t in this case is $t\leq T+B$. So we require $A+U\leq T+B$, or $U-T\leq B-A$.

$A+U\leq T+B$ In this case we have

$$\begin{aligned}\gamma_1 &= \inf_{A\leq s\leq t-U}\{f(t-s)+g(s)\}\\ &= \inf_{A\leq s\leq t-U}\{n\cdot(t-s-U)+m(U-T)+a(s-A)\}\end{aligned}$$

We have the following subcases

a) $a\geq n$
Then $a-n\geq 0$ and in order to minimize the linear function we have to take the lower bound, i.e. $s=A$, yielding

$$\gamma_1=n\cdot(t-(A+U))+m(U-T)$$

b) $a<n$
Then $a-n<0$ and in order to minimize the linear function we have to take the upper bound, i.e. $s=t-U$, yielding

$$\gamma_1=a\cdot(t-(A+U)+m(U-T))$$

So we have

$$\gamma_1=\min\{a,n\}\cdot(t-(A+U))+m(U-T)$$

For γ_2 we proceed in a similar manner.

$$\begin{aligned}\gamma_2 &= \inf_{t-U\leq s\leq t-T}\{f(t-s)+g(s)\}\\ &= \inf_{t-U\leq s\leq t-T}\{m(t-s-T)+a(s-A)\}\end{aligned}$$

We have the following subcases

a) $a\geq m$
Then $a-m\geq 0$ and in order to minimize the linear function we have to take the lower bound, i.e. $s=t-U$, yielding

$$\gamma_2=a\cdot(t-(A+U))+m(U-T)$$

b) $a < m$
Then $a - m < 0$ and in order to minimize the linear function we have to take the upper bound, i.e. $s = t - T$, yielding

$$\gamma_2 = a \cdot (t - (A + T))$$

Since $a \cdot (t - (A+T)) = a \cdot (t - (A+U)) + a(U - T)$, we have

$$\gamma_2 = a \cdot (t - (A + U)) + \min\{a, m\} \cdot (U - T)$$

So we have

$$\Gamma = \min\{\gamma_1, \gamma_2\}$$

$A + U > T + B$ In this case the expression for γ_1 does not make sense. Thus we only consider Γ

$$\begin{aligned} \Gamma &= \inf_{A \le s \le t-T} \{f(t-s) + g(s)\} \\ &= \inf_{A \le s \le t-T} \{m(t - s - T) + a(s - A)\} \end{aligned}$$

We have the following subcases

a) $a \geq m$
Then $a - m \geq 0$ and in order to minimize the linear function we have to take the lower bound, i.e. $s = A$, yielding

$$\gamma_2 = m \cdot (t - (A + T))$$

b) $a < m$
Then $a - m < 0$ and in order to minimize the linear function we have to take the upper bound, i.e. $s = t - T$, yielding

$$\gamma_2 = a \cdot (t - (A + T))$$

So we have

$$\Gamma = \min\{a, m\} \cdot (t - (A + T))$$

Step 3: $B \leq t - T$

$$\begin{aligned} (f * g)(t) &= \inf_{0 \le s \le t} \{f(t-s) + g(s)\} \\ &= \min\{\inf_{0 \le s \le t-T} \{f(t-s) + g(s)\}, \inf_{t-T \le s \le t} \{f(t-s) + g(s)\}\} \\ &= \min\{\inf_{0 \le s \le A} \{f(t-s) + g(s)\}, \inf_{A \le s \le t-T} \{f(t-s) + g(s)\}, \inf_{t-T \le s \le t} \{g(s)\}\} \\ &= \min\{f(t-A), \inf_{A \le s \le t-T} \{f(t-s) + g(s)\}, g(t-T)\} \\ &= \min\{f(t-A), \inf_{A \le s \le B} \{f(t-s) + g(s)\}, \\ &\qquad \inf_{B \le s \le t-T} \{f(t-s) + g(s)\}, b \cdot (t - (T + B)) + a \cdot (B - A)\} \end{aligned}$$

where we have used the monotonicity of both f and g.
We have

$$f(t-A) = \begin{cases} m \cdot (t-(T+A)) & T+A \leq t \leq U+A \\ n \cdot (t-(A+U)) + m \cdot (U-T) & U+A \leq t \end{cases}$$

Obviously we can also write

$$f(t-A) = \min\{m \cdot (t-(T+A)), n \cdot (t-(A+U)) + m \cdot (U-T)\}$$

We set

$$\Gamma_1 := \inf_{A \leq s \leq B} \{f(t-s) + g(s)\}$$

and

$$\Gamma_2 := \inf_{B \leq s \leq t-T} \{f(t-s) + g(s)\}$$

Then, clearly

$$(f * g)(t) = \min\{f(t-A), b \cdot (t-(T+B)) + a \cdot (B-A), \Gamma_1, \Gamma_2\}$$

Note, that by $U > T$ we have $-U < -T$ and thus $t-U < t-T$. In order to compute Γ_1 and Γ_2, we have to consider the following two possibilities: $t-U \leq B$ and $t-U > B$, since at $t-s = U$ f changes its behavior.

$t-U \leq B$ In this case we have to split up Γ_1 into γ_1 and γ_2:

$$\begin{aligned} \gamma_1 &= \inf_{A \leq s \leq t-U} \{f(t-s) + g(s)\} \\ &= \inf_{A \leq s \leq t-U} \{n \cdot (t-s-U) + m(U-T) + a(s-A)\} \\ \gamma_2 &= \inf_{t-U \leq s \leq B} \{f(t-s) + g(s)\} \\ &= \inf_{t-U \leq s \leq B} \{m \cdot (t-s-T) + a(s-A)\} \end{aligned}$$

We have to compute γ_1 and γ_2 as above. We will start with γ_1.

$a \geq n$ Then $a-n \geq 0$ and in order to minimize the linear function we have to take the lower bound, i.e. $s = A$, yielding

$$\gamma_1 = n \cdot (t-(A+U)) + m(U-T)$$

$a < n$ Then $a-n < 0$ and in order to minimize the linear function we have to take the upper bound, i.e. $s = t-U$, yielding

$$\gamma_1 = a \cdot (t-(A+U)) + m(U-T)$$

So we have

$$\gamma_1 = \min\{a, n\} \cdot (t-(A+U)) + m(U-T)$$

Similarly we get for γ_2

$a \geq m$ $a - m \geq 0$ and in order to minimize the linear function we have to take the lower bound, i.e. $s = t - U$, yielding

$$\gamma_2 = a \cdot (t - (A + U)) + m(U - T)$$

$a < m$ $a - m < 0$ and in order to minimize the linear function we have to take the upper bound, i.e. $s = B$, yielding

$$\gamma_2 = m \cdot (t - (T + B)) + a(B - A)$$

Further we have

$$\begin{aligned}\Gamma_2 &= \inf_{B \leq s \leq t-T} \{f(t-s) + g(s)\} \\ &= \inf_{B \leq s \leq t-T} \{m \cdot (t - s - T) + b(s - B) + a(B - A)\}\end{aligned}$$

As usual, we have two different cases:

$b \geq m$ Then $b - m \geq 0$ and in order to minimize the linear function we have to take the lower bound, i.e. $s = B$, yielding

$$\Gamma_2 = m \cdot (t - (T + B)) + a(B - A)$$

$b < m$ Then $b - m < 0$ and in order to minimize the linear function we have to take the upper bound, i.e. $s = t - T$, yielding

$$\Gamma_2 = b \cdot (t - (T + B)) + a(B - A)$$

So we have

$$\Gamma_2 = \min\{b, m\} \cdot (t - (T + B)) + a(B - A)$$

We have to determine the minimum of $\gamma_1, \gamma_2, \Gamma_2$ in order to compute the min–plus convolution. This will be done below, since this heavily depends on the choices of the parameters relative to each other.

$t - U > B$ We have for Γ_1

$$\begin{aligned}\Gamma_1 &= \inf_{A \leq s \leq B} \{f(t-s) + g(s)\} \\ &= \inf_{A \leq s \leq B} \{n(t - s - U) + m(U - T) + a(s - A)\}\end{aligned}$$

As usual, we have two different cases:

$a \geq n$ Then $a - n \geq 0$ and in order to minimize the linear function we have to take the lower bound, i.e. $s = A$, yielding

$$\Gamma_1 = n \cdot (t - (A + U)) + m(U - T)$$

$a < n$ Then $a - n < 0$ and in order to minimize the linear function we have to take the upper bound, i.e. $s = B$, yielding

$$\Gamma_1 = n \cdot (t - (B + U)) + m(U - T) + a(B - A)$$

So we have

$$\Gamma_1 = n \cdot (t - (B + U)) + \min\{a, n\}(B - A) + m(U - T)$$

In this case we have to split up Γ_2 into η_1 and η_2:

$$\begin{aligned}
\eta_1 &= \inf_{B \le s \le t-U} \{f(t-s) + g(s)\} \\
&= \inf_{B \le s \le t-U} \{n \cdot (t - s - U) + m(U - T) + b(s - B) + a(B - A)\} \\
\eta_2 &= \inf_{B \le s \le t-T} \{f(t-s) + g(s)\} \\
&= \inf_{t-U \le s \le t-T} \{m \cdot (t - s - T) + b(s - B) + a(B - A)\}
\end{aligned}$$

We have to compute η_1 and η_2 as above. We will start with η_1.

$b \ge n$ Then $b - n \ge 0$ and in order to minimize the linear function we have to take the lower bound, i.e. $s = B$, yielding

$$\eta_1 = n \cdot (t - (B + U)) + m(U - T) + a(B - A)$$

$b < n$ Then $b - n < 0$ and in order to minimize the linear function we have to take the upper bound, i.e. $s = t - U$, yielding

$$\eta_1 = b \cdot (t - (B + U)) + m(U - T) + a(B - A)$$

So we have

$$\eta_1 = \min\{b, n\} \cdot (t - (B + U)) + m(U - T) + a(B - A)$$

Similarly we get for η_2

$b \ge m$ $b - m \ge 0$ and in order to minimize the linear function we have to take the lower bound, i.e. $s = t - U$, yielding

$$\eta_2 = b \cdot (t - (B + U)) + m(U - T) + a(B - A)$$

$b < m$ $b - m < 0$ and in order to minimize the linear function we have to take the upper bound, i.e. $s = t - T$, yielding

$$\eta_2 = b \cdot (t - (B + T)) + a(B - A)$$

We have to determine the minimum of Γ_1, η_1, η_2 in order to compute the min–plus convolution. This will be done below, since this heavily depends on the choices of the parameters relative to each other.

Basically we now have everything to compute the min–plus convolution as the pointwise minimum of the affine functions appearing in the above calculations.

STEP II

Now we just have to collect the appropriate pieces of information. Therefore we will distinguish " various " cases.

$$a \geq b \geq m \geq n \qquad B - A \geq U - T$$

$t \leq T + A$ We have

$$(f * g)(t) = 0$$

$T + A < t \leq T + B$ We have

$$(f * g)(t) = \min\{f(t - A), \gamma_1(t), \gamma_2(t), a \cdot (t - (T + A))\}$$

with

$$\begin{aligned}
f(t-A) &= \begin{cases} m \cdot (t - (T + A)) & T + A \leq t \leq U + A \\ n \cdot (t - (A + U)) + m \cdot (U - T) & U + A \leq t \leq T + B \end{cases} \\
\gamma_1(t) &= n \cdot (t - (A + U)) + m(U - T) \\
\gamma_2(t) &= a \cdot (t - (A + U)) + m(U - T)
\end{aligned}$$

Obviously we have $\gamma_2(t) \geq \gamma_1(t)$ by $a \geq n$. First we have to decide for $T + A \leq t \leq U + A$, if $f(t - A) \leq \gamma_1(t)$.

$$\begin{aligned}
m \cdot (t - (T + A)) &\leq n \cdot (t - (A + U)) + m(U - T) \\
(m - n) \cdot t &\leq -n \cdot (A + U) + m(U + A) \\
t &\leq A + U
\end{aligned}$$

Looking at the definition of $f(t - A)$, we realize that this computation was superfluous, Now we consider $f(t - A) \leq a \cdot (t - (T + A))$ for $T + A \leq t \leq U + A$.

$$\begin{aligned}
m \cdot (t - (T + A)) &\leq a \cdot (t - (T + A)) \\
m &\leq a
\end{aligned}$$

which holds in this case. Finally we have to consider $f(t - A) \leq a \cdot (t - (T + A))$ for $U + A \leq t \leq T + B$.

$$\begin{aligned}
n \cdot (t - (A + U)) + m(U - T) &\leq a \cdot (t - (T + A)) \\
(n - a) \cdot t &\leq A(n - a) - T(a - m) + U(n - m) \\
t &\geq A + \frac{U(n - m) - T(a - m)}{n - a}
\end{aligned}$$

since $a \geq n$. We set

$$P := \frac{U(n - m) - T(a - m)}{n - a}$$

We have to answer the question, if $P \le U$, since then $A + P \le A + U$.

$$\begin{aligned}
\frac{U(n-m) - T(a-m)}{n-a} &\le U \\
U(n-m) - T(a-m) &\ge U(n-a) \\
(U-T)(a-m) &\ge 0
\end{aligned}$$

Since $U > T$ and $a \ge m$ this is satisfied. So we have

$$(f * g)(t) = \begin{cases} m \cdot (t - (T + A)) & T + A \le t \le U + A \\ n \cdot (t - (A + U)) + m \cdot (U - T) & U + A \le t \le T + B \end{cases}$$

$T + B < t \le U + B$ We have

$$(f * g)(t) = \min\{f(t - A), b \cdot (t - (T + B)) + a \cdot (B - A), \gamma_1, \gamma_2, \Gamma_2\}$$

with

$$\begin{aligned}
f(t - A) &= n \cdot (t - (A + U)) + m(U - T) \\
\gamma_1 &= n \cdot (t - (A + U)) + m(U - T) \\
\gamma_2 &= a \cdot (t - (A + U)) + m(U - T) \\
\Gamma_2 &= m \cdot (t - (T + B)) + a(B - A)
\end{aligned}$$

Clearly, $\gamma_1 \le \gamma_2$ and $\Gamma_2 \le b \cdot (t - (T + B)) + a \cdot (B - A)$. We just have to compute $\gamma_1(t) \le \Gamma_2(t)$

$$\begin{aligned}
n \cdot (t - (A + U)) + m(U - T) &\le m \cdot (t - (T + B)) + a(B - A) \\
(n - m) \cdot t &\le A(n - a) + B(a - m) + U(n - m) \\
t &\ge U + \frac{A(n-a) + B(a-m)}{n-m}
\end{aligned}$$

We set

$$Q := \frac{A(n-a) + B(a-m)}{n-m}$$

We are interested, if $Q \ge B$, since then $t \ge U + Q \ge U + B$ and hence not relevant.

$$\begin{aligned}
\frac{A(n-a) + B(a-m)}{n-m} &\ge B \\
A(n-a) + B(a-m) &\le B(n-m) \\
(B - A) \cdot (a - n) &\le 0
\end{aligned}$$

This is false, since $a \ge n$. So we have to decide, if $U + Q \le B + T$.

$$\begin{aligned}
U + \frac{A(n-a) + B(a-m)}{n-m} &\le B + T \\
U(n-m) + A(n-a) + B(a-m) &\ge (B + T)(n - m) \\
n(U + A - B - T) + m(T - U) + a(B - A) &\ge 0 \\
(a - n)(B - A) + (n - m)(U - T) &\ge 0 \qquad (4.2)
\end{aligned}$$

Now we use $a \geq n$ and $B - A \geq U - T$ to obtain

$$\begin{aligned}(a-n)(B-A)+(n-m)(U-T) &\geq (a-n)(U-T)+(n-m)(U-T) \\ &= (a-m)(U-T)\end{aligned}$$

Since $a \geq m$ we have shown (4.2). That is,

$$(f * g)(t) = n \cdot (t - (A+U)) + m(U-T)$$

$U + B < t$ We have

$$(f * g)(t) = \min\{f(t-A), b \cdot (t-(T+B)) + a \cdot (B-A), \Gamma_1, \eta_1, \eta_2\}$$

with

$$\begin{aligned}f(t-A) &= n \cdot (t-(A+U)) + m(U-T) \\ \Gamma_1 &= n \cdot (t-(A+U)) + m(U-T) \\ \eta_1 &= n \cdot (t-(B+U)) + m(U-T) + a(B-A) \\ \eta_2 &= b \cdot (t-(B+U)) + m(U-T) + a(B-A)\end{aligned}$$

Clearly we have $\eta_1 \leq \eta_2$. We have to decide, for which t $\Gamma_1 \leq \eta_1$ holds.

$$\begin{aligned}n \cdot (t-(A+U)) + m(U-T) &\leq n \cdot (t-(B+U)) + m(U-T) + a(B-A) \\ 0 &\leq n(A-B) + a(B-A) \\ 0 &\leq (a-n)(B-A)\end{aligned}$$

which is true in this case. We finally look at $n \cdot (t-(A+U)) + m(U-T) \leq b \cdot (t-(T+B)) + a \cdot (B-A)$.

$$\begin{aligned}n \cdot (t-(A+U)) + m(U-T) &\leq b \cdot (t-(T+B)) + a \cdot (B-A) \\ (n-b) \cdot t &\leq A(n-a) + B(a-b) + U(n-m) + T(m-b) \\ t &\geq \frac{A(n-a) + B(a-b) + U(n-m) + T(m-b)}{n-b}\end{aligned}$$

Setting

$$R := \frac{A(n-a) + B(a-b) + U(n-m) + T(m-b)}{n-b}$$

we have to decide, if $R \leq U + B$.

$$\begin{aligned}\frac{A(n-a) + B(a-b) + U(n-m) + T(m-b)}{n-b} &\leq U + B \\ A(n-a) + B(a-b) + U(n-m) + T(m-b) &\geq (U+B)(n-b) \\ m(T-U) + n(A-B) + a(B-A) + b(U-T) &\geq 0 \\ (a-n)(B-A) + (b-m)(U-T) &\geq 0 \qquad (4.3)\end{aligned}$$

We can show (4.3), since $B-A \geq U-T$ and $a-n \geq 0$. We get

$$\begin{aligned}(a-n)(B-A)+(b-m)(U-T) &\geq (a-n)(U-T)+(b-m)(U-T)\\ &= ((a-m)+(b-n))(U-T)\\ &\geq 0\end{aligned}$$

since we have $a \geq m$ and $b \geq n$! This yields

$$(f*g)(t) = n \cdot (t-(A+U)) + m(U-T)$$

Now we collect the results in one final expression, obtaining

$$(f*g)(t) = \begin{cases} 0 & t \leq T+A \\ m(t-(T+A)) & T+A < t \leq U+A \\ n \cdot (t-(A+U)) + m(U-T) & U+A < t \end{cases}$$

$a \geq b \geq m \geq n \qquad B-A < U-T$

$t \leq T+A$ We have

$$(f*g)(t) = 0$$

$T+A < t \leq T+B$ We have

$$(f*g)(t) = \min\{f(t-A), \Gamma(t), a \cdot (t-(T+A))\}$$

with

$$\begin{aligned} f(t-A) &= m \cdot (t-(T+A)) \\ \Gamma(t) &= m \cdot (t-(T+A)) \end{aligned}$$

Since $a \geq m$ $\Gamma(t) \leq a \cdot (t-(T+A))$ holds. So we have

$$(f*g)(t) = m \cdot (t-(T+A))$$

$T+B < t \leq U+B$ We have

$$(f*g)(t) = \min\{f(t-A), b \cdot (t-(T+B)) + a \cdot (B-A), \gamma_1, \gamma_2, \Gamma_2\}$$

with

$$\begin{aligned} f(t-A) &= \begin{cases} m \cdot (t-(T+A)) & T+A \leq t \leq U+A \\ n \cdot (t-(A+U)) + m \cdot (U-T) & U+A \leq t \end{cases} \\ \gamma_1 &= n \cdot (t-(A+U)) + m(U-T) \\ \gamma_2 &= a \cdot (t-(A+U)) + m(U-T) \\ \Gamma_2 &= m \cdot (t-(T+B)) + a(B-A) \end{aligned}$$

Since $A < B$, we have $T + A < T + B$ and further $U + A < U + B$. So f changes in this case. Since $a \geq n$, we have $\gamma_1 \leq \gamma_2$. As above one computes $m \cdot (t - (T + A)) \leq n \cdot (t - (A + U)) + m \cdot (U - T)$ iff $t \leq A + U$.
Now let $B+T \leq t \leq A+U$. Here we have $f(t-A) \leq \gamma_1$. We check if $f(t-A) \leq \Gamma_2$.

$$\begin{aligned} m \cdot (t - (T + A)) &\leq m \cdot (t - (T + B)) + a(B - A) \\ 0 &\leq m(A - B) + a(B - A) \\ 0 &\leq (B - A)(a - m) \end{aligned}$$

which is true in this case. We also check $f(t - A) \leq b \cdot (t - (T + B)) + a \cdot (B - A)$.

$$\begin{aligned} m \cdot (t - (T + A)) &\leq b \cdot (t - (T + B)) + a \cdot (B - A) \\ (m - b) \cdot t &\leq T(m - b) + B(a - b) + A(m - a) \\ t &\geq T + \frac{B(a - b) + A(m - a)}{m - b} \end{aligned}$$

We set

$$S := \frac{B(a - b) + A(m - a)}{m - b}$$

and want to know if $S \leq B$, since then $T + S \leq T + B$.

$$\begin{aligned} \frac{B(a - b) + A(m - a)}{m - b} &\leq B \\ B(a - b) + A(m - a) &\geq B(m - b) \\ (a - m)(B - A) &\geq 0 \end{aligned}$$

This is true, so we have

$$(f * g)(t) = m \cdot (t - (T + A)) \qquad B + T \leq t \leq A + U$$

Now we investigate the case $A + U < t \leq B + U$. Here we have $f(t - A) = \gamma_1$. We have to check if $\gamma_1 \leq \Gamma_2$.

$$\begin{aligned} n \cdot (t - (A + U)) + m(U - T) &\leq m \cdot (t - (T + B)) + a(B - A) \\ t &\geq U + Q \end{aligned}$$

with

$$Q := \frac{A(n - a) + B(a - m)}{n - m}$$

We already showed above, that $U + Q \leq U + B$. So we have to decide, if $U + A \leq U + Q$, i.e. $A \leq Q$.

$$\begin{aligned} A &\leq \frac{A(n - a) + B(a - m)}{n - m} \\ 0 &\geq A(m - a) + B(a - m) \\ 0 &\geq (B - A)(a - m) \end{aligned}$$

which is false. Hence it is left to check if $\gamma_1 \leq b \cdot (t - (T + B)) + a \cdot (B - A)$. But clearly we have

$$m \cdot (t - (T + B)) + a \cdot (B - A) \leq b \cdot (t - (T + B)) + a \cdot (B - A)$$

and thus

$$(f * g)(t) = n \cdot (t - (A + U)) + m(U - T) \qquad U + A \leq t \leq U + B$$

$U + B < t$ We have

$$(f * g)(t) = \min\{f(t - A), b \cdot (t - (T + B)) + a \cdot (B - A), \Gamma_1, \eta_1, \eta_2\}$$

with

$$\begin{aligned} f(t - A) &= n \cdot (t - (A + U)) + m \cdot (U - T) \\ \Gamma_1 &= n \cdot (t - (A + U)) + m(U - T) \\ \eta_1 &= n \cdot (t - (B + U)) + m(U - T) + a(B - A) \\ \eta_2 &= b \cdot (t - (B + U)) + m(U - T) + a(B - A) \end{aligned}$$

We have $\eta_1 \leq \eta_2$. We have to decide when $\Gamma_1 \leq \eta_1$.

$$\begin{aligned} n \cdot (t - (A + U)) + m(U - T) &\leq n \cdot (t - (B + U)) + m(U - T) + a(B - A) \\ 0 &\leq a(B - A) + n(A - B) \\ 0 &\leq (B - A)(a - n) \end{aligned}$$

This holds true for this parameter set. It remains to consider $\Gamma_1 \leq b \cdot (t - (T + B)) + a \cdot (B - A)$.

$$\begin{aligned} n \cdot (t - (A + U)) + m(U - T) &\leq b \cdot (t - (T + B)) + a \cdot (B - A) \\ t &\geq R \end{aligned}$$

with

$$R := \frac{A(n - a) + B(a - b) + U(n - m) + T(m - b)}{n - b}$$

we have to decide, if $R \leq U + B$.

$$\begin{aligned} \frac{A(n - a) + B(a - b) + U(n - m) + T(m - b)}{n - b} &\leq U + B \\ A(n - a) + B(a - b) + U(n - m) + T(m - b) &\geq (U + B)(n - b) \\ m(T - U) + n(A - B) + a(B - A) + b(U - T) &\geq 0 \\ (a - n)(B - A) + (b - m)(U - T) &\geq 0 \end{aligned} \tag{4.4}$$

We can show (4.4); since $B - A < U - T$ and $b - m \geq 0$. We get

$$\begin{aligned} (a - n)(B - A) + (b - m)(U - T) &\geq (a - n)(B - A) + (b - m)(B - A) \\ &= ((a - m) + (b - n))(B - A) \\ &\geq 0 \end{aligned}$$

Hence we have

$$(f * g)(t) = n \cdot (t - (A + U)) + m(U - T) \qquad U + b < T$$

Collecting the above results we have for this case

$$(f * g)(t) = \begin{cases} 0 & t \leq T + A \\ m(t - (T + A)) & T + A < t \leq U + A \\ n \cdot (t - (A + U)) + m(U - T) & U + A < t \end{cases}$$

This is exactly the same transform as in the first case!

$a \geq m > b \geq n \qquad B - A \geq U - T$

$t \leq T + A$ We have

$$(f * g)(t) = 0$$

$T + A < t \leq T + B$ We have

$$(f * g)(t) = \min\{f(t - A), \gamma_1(t), \gamma_2(t), a \cdot (t - (T + A))\}$$

with

$$\begin{aligned} f(t - A) &= \begin{cases} m \cdot (t - (T + A)) & T + A \leq t \leq U + A \\ n \cdot (t - (A + U)) + m \cdot (U - T) & U + A \leq t \leq T + B \end{cases} \\ \gamma_1(t) &= n \cdot (t - (A + U)) + m(U - T) \\ \gamma_2(t) &= a \cdot (t - (A + U)) + m(U - T) \end{aligned}$$

The whole argumentation now is analogous to the one in the case $a \geq b \geq m \geq n$, since for the t in question we have only used $U > T$, $a \geq m$ and $a \geq n$. So we have

$$(f * g)(t) = \begin{cases} m \cdot (t - (T + A)) & T + A \leq t \leq U + A \\ n \cdot (t - (A + U)) + m \cdot (U - T) & U + A \leq t \leq T + B \end{cases}$$

$T + B < t \leq U + B$ We have

$$(f * g)(t) = \min\{f(t - A), b \cdot (t - (T + B)) + a \cdot (B - A), \gamma_1, \gamma_2, \Gamma_2\}$$

with

$$\begin{aligned} f(t - A) &= n \cdot (t - (A + U)) + m(U - T) \\ \gamma_1 &= n \cdot (t - (A + U)) + m(U - T) \\ \gamma_2 &= a \cdot (t - (A + U)) + m(U - T) \\ \Gamma_2 &= b \cdot (t - (T + B)) + a(B - A) \end{aligned}$$

Note that now Γ_2 is different from the one encountered in the case $a \geq b \geq m \geq n$. Clearly, we have $\gamma_1 \leq \gamma_2$, since $a \geq n$. So it remains to consider

$$\begin{aligned} n \cdot (t - (A+U)) + m(U-T) &\leq b \cdot (t - (T+B)) + a(B-A) \\ (n-b) \cdot t &\leq A(n-a) + B(a-b) + T(m-b) + U(n-m) \\ t &\geq \frac{A(n-a) + B(a-b) + T(m-b) + U(n-m)}{n-b} \end{aligned}$$

where we used $b \geq n$. Setting

$$K := \frac{A(n-a) + B(a-b) + T(m-b) + U(n-m)}{n-b}$$

we have to decide, if $K \leq T + B$.

$$\begin{aligned} A(n-a) + B(a-b) + T(m-b) + U(n-m) &\geq (T+B)(n-b) \\ a(B-A) + m(T-U) + n(A+U-T-B) &\geq 0 \\ (a-n)(B-A) + (n-m)(U-T) &\geq 0 \end{aligned} \tag{4.5}$$

Since $a \geq n$ and $B - A \geq (U - T)$ we can further estimate (4.5)

$$\begin{aligned} (a-n)(B-A) + (n-m)(U-T) &\geq (a-m)(U-T) \\ &\geq 0 \end{aligned}$$

since $a \geq m$ and $U > T$, thus showing $K \leq T + B$. So we have

$$(f * g)(t) = n \cdot (t - (A+U)) + m(U-T)$$

$U + B < t$ We have

$$(f * g)(t) = \min\{f(t-A), b \cdot (t - (T+B)) + a \cdot (B-A), \Gamma_1, \eta_1, \eta_2\}$$

with

$$\begin{aligned} f(t-A) &= n \cdot (t - (A+U)) + m(U-T) \\ \Gamma_1 &= n \cdot (t - (A+U)) + m(U-T) \\ \eta_1 &= n \cdot (t - (B+U)) + m(U-T) + a(B-A) \\ \eta_2 &= b \cdot (t - (T+B)) + a(B-A) \end{aligned}$$

Note that only η_2 changed comparing to the case $a \geq b \geq m \geq n$. We have to decide, for which t $\Gamma_1 \leq \eta_1$ holds.

$$\begin{aligned} n \cdot (t - (A+U)) + m(U-T) &\leq n \cdot (t - (B+U)) + m(U-T) + a(B-A) \\ 0 &\leq n(A-B) + a(B-A) \\ 0 &\leq (a-n)(B-A) \end{aligned}$$

which is true in this case. We finally look at

$$\begin{aligned}
n\cdot(t-(A+U))+m(U-T) &\leq b\cdot(t-(T+B))+a\cdot(B-A)\\
(n-b)\cdot t &\leq A(n-a)+B(a-b)+U(n-m)+T(m-b)\\
t &\geq \frac{A(n-a)+B(a-b)+U(n-m)+T(m-b)}{n-b}
\end{aligned}$$

Setting

$$R := \frac{A(n-a)+B(a-b)+U(n-m)+T(m-b)}{n-b}$$

we have to decide, if $R \leq U+B$.

$$\begin{aligned}
\frac{A(n-a)+B(a-b)+U(n-m)+T(m-b)}{n-b} &\leq U+B\\
A(n-a)+B(a-b)+U(n-m)+T(m-b) &\geq (U+B)(n-b)\\
m(T-U)+n(A-B)+a(B-A)+b(U-T) &\geq 0\\
(a-n)(B-A)+(b-m)(U-T) &\geq 0 \qquad (4.6)
\end{aligned}$$

We can show (4.6), since $B-A \geq U-T$ and $a-n \geq 0$. We get

$$\begin{aligned}
(a-n)(B-A)+(b-m)(U-T) &\geq (a-n)(U-T)+(b-m)(U-T)\\
&= ((a-m)+(b-n))(U-T)\\
&\geq 0
\end{aligned}$$

since we have $a \geq m$ and $b \geq n$! This yields

$$(f*g)(t) = n\cdot(t-(A+U))+m(U-T)$$

Now we collect the results in one final expression, obtaining

$$(f*g)(t) = \begin{cases} 0 & t \leq T+A\\ m(t-(T+A)) & T+A < t \leq U+A\\ n\cdot(t-(A+U))+m(U-T) & U+A < t \end{cases}$$

$a \geq m > b \geq n \qquad B-A < U-T$

$t \leq T+A$ We have

$$(f*g)(t) = 0$$

$T+A < t \leq T+B$ We have

$$(f*g)(t) = \min\{f(t-A), \Gamma(t), a\cdot(t-(T+A))\}$$

with

$$\begin{aligned} f(t-A) &= m\cdot(t-(T+A)) \\ \Gamma(t) &= m\cdot(t-(T+A)) \end{aligned}$$

Since $a \geq m$ $\Gamma(t) \leq a\cdot(t-(T+A))$ holds. So we have

$$(f*g)(t) = m\cdot(t-(T+A))$$

$T+B < t \leq U+B$ We have

$$(f*g)(t) = \min\{f(t-A), b\cdot(t-(T+B)) + a\cdot(B-A), \gamma_1, \gamma_2, \Gamma_2\}$$

with

$$\begin{aligned} f(t-A) &= \begin{cases} m\cdot(t-(T+A)) & T+A \leq t \leq U+A \\ n\cdot(t-(A+U)) + m\cdot(U-T) & U+A \leq t \end{cases} \\ \gamma_1 &= n\cdot(t-(A+U)) + m(U-T) \\ \gamma_2 &= a\cdot(t-(A+U)) + m(U-T) \\ \Gamma_2 &= b\cdot(t-(T+B)) + a(B-A) \end{aligned}$$

Note that only Γ_2 changed comparing to the situation for $a \geq b \geq m \geq n$.
Since $A < B$, we have $T+A < T+B$ and further $U+A < U+B$. So f changes in this case. Since $a \geq n$, we have $\gamma_1 \leq \gamma_2$. As above one computes $m\cdot(t-(T+A)) \leq n\cdot(t-(A+U)) + m\cdot(U-T)$ iff $t \leq A+U$.
Now let $B+T \leq t \leq A+U$. Here we have $f(t-A) \leq \gamma_1$. We check if $f(t-A) \leq \Gamma_2$.

$$\begin{aligned} m\cdot(t-(T+A)) &\leq b\cdot(t-(T+B)) + a\cdot(B-A) \\ (m-b)\cdot t &\leq T(m-b) + B(a-b) + A(m-a) \\ t &\leq T + \frac{B(a-b)+A(m-a)}{m-b} \end{aligned}$$

We set

$$S := \frac{B(a-b)+A(m-a)}{m-b}$$

and want to know if $S \geq B$, since then $T+S \geq T+B$.

$$\begin{aligned} \frac{B(a-b)+A(m-a)}{m-b} &\geq B \\ B(a-b)+A(m-a) &\geq B(m-b) \\ (a-m)(B-A) &\geq 0 \end{aligned}$$

This is true since $a \geq m$ and $B > A$. We now have to decide if $T+S \leq A+U$.

$$\begin{aligned} T(m-b) + B(a-b) + A(m-a) &\leq (A+U)(m-b) \\ a(B-A) + b(-T-B+U+A) + m(T-U) &\leq 0 \\ (a-b)(B-A) + (b-m)(U-T) &\leq 0 \end{aligned} \tag{4.7}$$

Since $b - m \leq 0$ and $a - b > 0$, we can not estimate this in general. We choose for example $U - T = k(B - A)$, $k \geq 1$, $a = 4$, $b = 1$ and $m = 3$. Then (4.7) becomes

$$(B - A)(3 - 2k)$$

Since $B > A$, we get different signs for appropriate choice of k. So we have the follwing subcases:

i)

$$(a - b)(B - A) + (b - m)(U - T) \leq 0$$

ii)

$$(a - b)(B - A) + (b - m)(U - T) > 0$$

For $A + U \leq t \leq U + B$ we have to decide, if

$$\begin{aligned} n \cdot (t - (A + U)) + m(U - T) &\leq b \cdot (t - (T + B)) + a(B - A) \\ (n - b) \cdot t &\leq T(m - b) + U(n - m) + B(a - b) + A(n - a) \\ t &\geq \frac{T(m - b) + U(n - m) + B(a - b) + A(n - a)}{n - b} \end{aligned}$$

We set

$$V := \frac{T(m - b) + U(n - m) + B(a - b) + A(n - a)}{n - b}$$

We have to decide, if $V \leq A + U$.

$$\begin{aligned} T(m - b) + U(n - m) + B(a - b) + A(n - a) &\geq (A + U)(n - b) \\ a(B - A) + b(-T + U + A - B) + m(T - U) &\geq 0 \\ (a - b)(B - A) + (a - m)(U - T) &\geq 0 \end{aligned}$$

This holds true, since $A > B, U > T, a > b$ and $a \geq m$.
We are also interested if $V \leq T + S$

$$\begin{aligned} \frac{T(m - b) + U(n - m) + B(a - b) + A(n - a)}{n - b} &\leq \\ T + \frac{B(a - b) + A(m - a)}{m - b} & \\ (T(m - b) + U(n - m) + B(a - b) + A(n - a))(m - b) &\geq \\ (T(m - b) + B(a - b) + A(m - a))(n - b) & \\ T(m - b)(m - n) + U(n - m)(m - b) + B(a - b)(m - n) & \\ +A((n - a)(m - b) - (m - a)(n - b)) &\geq 0 \\ (U - T)(n - m)(m - b) + B(a - b)(m - n) + A(a - b)(n - m) &\geq 0 \\ (U - T)(n - m)(m - b) + (B - A)(a - b)(m - n) &\geq 0 \\ (n - m)\left((U - T)(m - b) + (B - A)(b - a)\right) &\geq 0 \\ (U - T)(m - b) + (B - A)(b - a) &\leq 0 \\ (B - A)(a - b) + (U - T)(b - m) &\geq 0 \end{aligned}$$

This is exactly (4.7) with the opposite inequality sign. So if we have $T+S \leq A+U$ then we get automatically $T+S \leq V \leq A+U$!
So if (4.7) **holds**, the min–plus convolution reads

$$(f*g)(t) = \begin{cases} m\cdot(t-(T+A)) & T+B \leq t \leq T+S \\ b\cdot(t-(T+B))+a\cdot(B-A) & T+S < t \leq V \\ n\cdot(t-(A+U))+m(U-T) & V < t \leq U+B \end{cases}$$

If (4.7) **is false**, we get instead

$$(f*g)(t) = \begin{cases} m\cdot(t-(T+A)) & T+B \leq t \leq A+U \\ n\cdot(t-(A+U))+m(U-T) & A+U < t \leq U+B \end{cases}$$

Remark

Remark 4.4 It follows from this behavior that it is possibly impossible to estimate the number of slopes occuring in the min–plus convolution beforehand. It heavily depends on the parameters. It is not clear if one can find a general formula like (4.7) in order to decide the number of slopes in the result.

$U+B<t$ We have

$$(f*g)(t) = \min\{f(t-A), b\cdot(t-(T+B))+a\cdot(B-A), \Gamma_1, \eta_1, \eta_2\}$$

with

$$\begin{aligned} f(t-A) &= n\cdot(t-(A+U))+m\cdot(U-T) \\ \Gamma_1 &= n\cdot(t-(A+U))+m(U-T) \\ \eta_1 &= n\cdot(t-(B+U))+m(U-T)+a(B-A) \\ \eta_2 &= b\cdot(t-(B+T))+a(B-A) \end{aligned}$$

Note that only η_2 changed comparing to the case $a \geq b \geq m \geq n$, $B-A < U-T$. We can rewrite Γ_1

$$n\cdot(t-(A+U))+m(U-T) = n\cdot(t-(B+U))+m(U-T)+n(B-A)$$

Since $a \geq n$ and $B > A$ we immediately realize $\Gamma_1 \leq \eta_1$. So we have to decide for which t we have $\Gamma_1(t) \leq \eta_2(t)$.

$$n\cdot(t-(A+U))+m(U-T) \leq b\cdot(t-(B+T))+a(B-A)$$

We have already done this in for $A+U \leq t \leq U+B$; we have $\Gamma_1(t) \leq \eta_1(t)$ iff $t \geq V$. Recall, $V \leq A+U$ and since $A+U \leq U+B$ we have independent of (4.7)

$$(f*g)(t) = n\cdot(t-(A+U))+m(U-T) \qquad U+B < t$$

So we collect the results:

$$a \geq m > b \geq n \qquad B - A < U - T$$
$$(a-b)(B-A) + (b-m)(U-T) \leq 0$$

$$(f * g)(t) = \begin{cases} 0 & t \leq T + A \\ m \cdot (t - (T+A)) & T + A \leq t \leq T + S \\ b \cdot (t - (T+B)) + a \cdot (B - A) & T + S < t \leq V \\ n \cdot (t - (A+U)) + m(U-T) & V < t \end{cases}$$

with

$$S := \frac{B(a-b) + A(m-a)}{m-b}$$
$$V := \frac{T(m-b) + U(n-m) + B(a-b) + A(n-a)}{n-b}$$

$$a \geq m > b \geq n \qquad B - A < U - T$$
$$(a-b)(B-A) + (b-m)(U-T) > 0$$

$$(f * g)(t) = \begin{cases} 0 & t \leq T + A \\ m \cdot (t - (T+A)) & T + A \leq t \leq A + U \\ n \cdot (t - (A+U)) + m(U-T) & A + U < t \end{cases}$$

$a \geq m \geq n > b \qquad B - A \geq U - T$

$t \leq T + A$ We have

$$(f * g)(t) = 0$$

$T + A < t \leq T + B$ We have

$$(f * g)(t) = \min\{f(t-A), \gamma_1(t), \gamma_2(t), a \cdot (t - (T+A))\}$$

with

$$f(t-A) = \begin{cases} m \cdot (t - (T+A)) & T + A \leq t \leq U + A \\ n \cdot (t - (A+U)) + m \cdot (U-T) & U + A \leq t \leq T + B \end{cases}$$
$$\gamma_1(t) = n \cdot (t - (A+U)) + m(U-T)$$
$$\gamma_2(t) = a \cdot (t - (A+U)) + m(U-T)$$

The whole argumentation now is analogous to the one in the case $a \geq b \geq m \geq n$, since for the t in question we have only used $U > T$, $a \geq m$ and $a \geq n$. So we have

$$(f * g)(t) = \begin{cases} m \cdot (t - (T+A)) & T + A \leq t \leq U + A \\ n \cdot (t - (A+U)) + m \cdot (U-T) & U + A \leq t \leq T + B \end{cases}$$

$T + B < t \leq U + B$ We have

$$(f * g)(t) = \min\{f(t-A), b \cdot (t - (T+B)) + a \cdot (B-A), \gamma_1, \gamma_2, \Gamma_2\}$$

with

$$\begin{aligned} f(t-A) &= n \cdot (t - (A+U)) + m(U-T) \\ \gamma_1 &= n \cdot (t - (A+U)) + m(U-T) \\ \gamma_2 &= a \cdot (t - (A+U)) + m(U-T) \\ \Gamma_2 &= b \cdot (t - (T+B)) + a(B-A) \end{aligned}$$

Since $a \geq n$, we have $\gamma_1 \leq \gamma_2$. So we have to decide, if $\gamma_1 \leq \Gamma_2$.

$$\begin{aligned} n \cdot (t - (A+U)) + m(U-T) &\leq b \cdot (t - (T+B)) + a(B-A) \\ (n-b) \cdot t &\leq T(m-b) + B(a-b) + A(n-a) + U(n-m) \\ t &\leq \frac{T(m-b) + B(a-b) + A(n-a) + U(n-m)}{n-b} \end{aligned}$$

where we used $n > b$. We set

$$M := \frac{T(m-b) + B(a-b) + A(n-a) + U(n-m)}{n-b}$$

First we have to estimate whether $T + B \leq M$.

$$\begin{aligned} (T+B)(n-b) &\leq T(m-b) + B(a-b) + A(n-a) + U(n-m) \\ 0 &\leq a(B-A) + m(T-U) + n(A-B+U-T) \\ 0 &\leq (B-A)(a-n) + (U-T)(n-m) \end{aligned}$$

Since $B - A \geq U - T$ and $a - n \geq 0$ we get

$$\begin{aligned} (B-A)(a-n) + (U-T)(n-m) &\geq (U-T)(a-n+n-m) \\ &= (U-T)(a-m) \end{aligned}$$

This last expression is nonnegative, thus we have $T + B \leq M$. Now we want to decide if $U + B \leq M$.

$$\begin{aligned} (U+B)(n-b) &\leq T(m-b) + B(a-b) + A(n-a) + U(n-m) \\ 0 &\leq a(B-A) + b(U-T) + m(T-U) + n(A-B) \\ 0 &\leq (a-n)(B-A) + (b-m)(U-T) \qquad (4.8) \end{aligned}$$

We have

$$\begin{aligned} (a-n)(B-A) + (b-m)(U-T) &\geq (a-n)(U-T) + (b-m)(U-T) \\ &= (a-n+b-m)(U-T) \end{aligned}$$

Here we again encounter the problem, that this can not be shown to be nonnegative in general. Take for example $b = 1, n = 2, m = 4, a = 4.1$. Further, with this choice and $(B - A) = k(U - T)$, $k \geq 1$, (4.8) becomes

$$(U - T)(2.1k - 3)$$

Since $U > T$, we get different signs for adequate choices of the parameter k. Hence we will have to distinguish in each case another set of cases, namely

i)

$$(a - n)(B - A) + (b - m)(U - T) \geq 0$$

We get

$$(f * g)(t) = \quad n \cdot (t - (A + U)) + m \cdot (U - T) \qquad T + B \; < \; t \; \leq \; U + B$$

ii)

$$(a - n)(B - A) + (b - m)(U - T) < 0$$

We get

$$(f * g)(t) = \begin{cases} n \cdot (t - (A + U)) + m(U - T) & T + B \leq t \leq M \\ b \cdot (t - (T + B)) + a(B - A) & M \leq t \leq U + B \end{cases}$$

Note that in the case $a \geq m > b \geq n, B - A \geq U - T$ we derived instead of (4.8):

$$(a - n)(B - A) + (n - m)(U - T) \geq 0$$

Now, since $b < n$ we get $b - m$ instead of $n - m$ and thus can not estimate the expression like we did before:

$$\begin{aligned} (a - n)(B - A) + (n - m)(U - T) &\geq (a - n + n - m)(U - T) \\ &= (a - m)(U - T) \qquad \geq 0 \end{aligned}$$

$U + B < t$ We have

$$(f * g)(t) = \min\{f(t - A), b \cdot (t - (T + B)) + a \cdot (B - A), \Gamma_1, \eta_1, \eta_2\}$$

with

$$\begin{aligned} f(t - A) &= n \cdot (t - (A + U)) + m(U - T) \\ \Gamma_1 &= n \cdot (t - (A + U)) + m(U - T) \\ \eta_1 &= b \cdot (t - (B + U)) + m(U - T) + a(B - A) \\ \eta_2 &= b \cdot (t - (B + T)) + a(B - A) \end{aligned}$$

We can rewrite η_2 and obtain

$$b \cdot (t - (B + T)) + a(B - A) = b \cdot (t - (B + U)) + b(U - T) + a(B - A)$$

Since $b < m$ we have $\eta_2 \leq \eta_1$. So we have to decide, for which t we have $\Gamma_1(t) \leq \eta_2(t)$. We have already computed this above. There are two cases: **If** (4.8) **holds**, then $U + B \leq M$ and we get

$$(f * g)(t) = \begin{cases} n \cdot (t - (A + U)) + m(U - T) & U + B \leq t \leq M \\ b \cdot (t - (T + B)) + a(B - A) & M < t \end{cases}$$

If (4.8) **is wrong**, then $m < U + B$ and we get

$$(f * g)(t) = \quad b \cdot (t - (T + B)) + a(B - A) \qquad U + B < t$$

So we collect the results:

$$a \geq m \geq n > b \qquad B - A \geq U - T$$
$$(a - n)(B - A) + (b - m)(U - T) \geq 0$$

$$(f * g)(t) = \begin{cases} 0 & t \leq T + A \\ m \cdot (t - (T + A)) & T + A \leq t \leq U + A \\ n \cdot (t - (A + U)) + m \cdot (U - T) & U + A < t \leq M \\ b \cdot (t - (T + B)) + a(B - A) & M < t \end{cases}$$

with

$$M := \frac{T(m - b) + B(a - b) + A(n - a) + U(n - m)}{n - b}$$

$$a \geq m \geq n > b \qquad B - A \geq U - T$$
$$(a - n)(B - A) + (b - m)(U - T) < 0$$

$$(f * g)(t) = \begin{cases} 0 & t \leq T + A \\ m \cdot (t - (T + A)) & T + A \leq t \leq U + A \\ n \cdot (t - (A + U)) + m(U - T) & U + A < t \leq M \\ b \cdot (t - (T + B)) + a(B - A) & M < t \end{cases}$$

with

$$M := \frac{T(m - b) + B(a - b) + A(n - a) + U(n - m)}{n - b}$$

Surprisingly one gets the same expression for the min–plus convolution! So the sign of $(a-n)(B-A)+(b-m)(U-T)$ plays only a role in deciding if $M \leq T+B$ or $M > T+B$. So we have the following result result

$$a \geq m \geq n > b \qquad B - A \geq U - T$$

$$(f * g)(t) = \begin{cases} 0 & t \leq T + A \\ m \cdot (t - (T + A)) & T + A \leq t \leq U + A \\ n \cdot (t - (A + U)) + m(U - T) & U + A < t \leq M \\ b \cdot (t - (T + B)) + a(B - A) & M < t \end{cases}$$

with

$$M := \frac{T(m-b) + B(a-b) + A(n-a) + U(n-m)}{n-b}$$

$a \geq m \geq n > b \qquad B - A < U - T$

$t \leq T + A$ We have

$$(f * g)(t) = 0$$

$T + A < t \leq T + B$ We have

$$(f * g)(t) = \min\{f(t-A), \Gamma(t), a \cdot (t - (T+A))\}$$

with

$$\begin{aligned} f(t-A) &= m \cdot (t - (T+A)) \\ \Gamma(t) &= m \cdot (t - (T+A)) \end{aligned}$$

Since $a \geq m$ $\Gamma(t) \leq a \cdot (t - (T+A))$ holds. So we have

$$(f * g)(t) = m \cdot (t - (T+A))$$

$T + B < t \leq U + B$ We have

$$(f * g)(t) = \min\{f(t-A), b \cdot (t - (T+B)) + a \cdot (B-A), \gamma_1, \gamma_2, \Gamma_2\}$$

with

$$\begin{aligned} f(t-A) &= \begin{cases} m \cdot (t - (T+A)) & T + A \leq t \leq U + A \\ n \cdot (t - (A+U)) + m \cdot (U-T) & U + A \leq t \end{cases} \\ \gamma_1 &= n \cdot (t - (A+U)) + m(U-T) \\ \gamma_2 &= a \cdot (t - (A+U)) + m(U-T) \\ \Gamma_2 &= b \cdot (t - (T+B)) + a(B-A) \end{aligned}$$

Since $A < B$, we have $T + A < T + B$ and further $U + A < U + B$. So f changes in this case. Since $a \geq n$, we have $\gamma_1 \leq \gamma_2$. As above one computes $m \cdot (t - (T+A)) \leq n \cdot (t - (A+U)) + m \cdot (U-T)$ iff $t \leq A + U$.
Now let $B+T \leq t \leq A+U$. Here we have $f(t-A) \leq \gamma_1$. We check if $f(t-A) \leq \Gamma_2$.

$$\begin{aligned} m \cdot (t - (T+A)) &\leq b \cdot (t - (T+B)) + a \cdot (B-A) \\ (m-b) \cdot t &\leq T(m-b) + B(a-b) + A(m-a) \\ t &\leq T + \frac{B(a-b) + A(m-a)}{m-b} \end{aligned}$$

We set

$$S := \frac{B(a-b) + A(m-a)}{m-b}$$

and want to know if $S \geq B$, since then $T + S \geq T + B$.

$$\begin{aligned} \frac{B(a-b) + A(m-a)}{m-b} &\geq B \\ B(a-b) + A(m-a) &\geq B(m-b) \\ (a-m)(B-A) &\geq 0 \end{aligned}$$

This is true since $a \geq m$ and $B > A$. We now have to decide if $T + S \leq A + U$.

$$\begin{aligned} T(m-b) + B(a-b) + A(m-a) &\leq (A+U)(m-b) \\ a(B-A) + b(-T-B+U+A) + m(T-U) &\leq 0 \\ (a-b)(B-A) + (b-m)(U-T) &\leq 0 \end{aligned} \quad (4.9)$$

Since $b - m \leq 0$ and $a - b > 0$, we can not estimate this in general. We choose for example $U - T = k(B - A)$, $k \geq 1$, $a = 4$, $b = 1$ and $m = 3$. Then (4.9) becomes

$$(B-A)(3-2k)$$

Since $B > A$, we get different signs for appropriate choice of k. So we have the follwing subcases:

i)

$$(a-b)(B-A) + (b-m)(U-T) \leq 0$$

ii)

$$(a-b)(B-A) + (b-m)(U-T) > 0$$

For $A + U \leq t \leq U + B$ we have to decide, if

$$\begin{aligned} n \cdot (t - (A+U)) + m(U-T) &\leq b \cdot (t - (T+B)) + a(B-A) \\ (n-b) \cdot t &\leq T(m-b) + U(n-m) + B(a-b) + A(n-a) \\ t &\leq \frac{T(m-b) + U(n-m) + B(a-b) + A(n-a)}{n-b} \end{aligned}$$

Note that in contrast to the case $a \geq m > b \geq n$, $B - A < U - T$ the inequality sign does not change since now $n - b > 0$! We set

$$V := \frac{T(m-b) + U(n-m) + B(a-b) + A(n-a)}{n-b}$$

We are interested if $V \leq U + B$.

$$\begin{aligned} T(m-b) + U(n-m) + B(a-b) + A(n-a) &\leq (U+B)(n-b) \\ a(B-A) + b(U-T) + m(T-U) + n(A-B) &\leq 0 \\ (a-n)(B-A) + (b-m)(U-T) &\leq 0 \end{aligned}$$

We have to decide, if $V \leq A + U$.

$$\begin{aligned} T(m-b) + U(n-m) + B(a-b) + A(n-a) &\leq (A+U)(n-b) \\ a(B-A) + b(-T+U+A-B) + m(T-U) &\leq 0 \\ (a-b)(B-A) + (b-m)(U-T) &\leq 0 \end{aligned}$$

This is again the condition (4.9).
We are also interested if $V \leq T + S$.

$$\begin{aligned} \frac{T(m-b) + U(n-m) + B(a-b) + A(n-a)}{n-b} &\leq \\ T + \frac{B(a-b) + A(m-a)}{m-b} & \\ (T(m-b) + U(n-m) + B(a-b) + A(n-a))(m-b) &\leq \\ (T(m-b) + B(a-b) + A(m-a))(n-b) & \\ T(m-b)(m-n) + U(n-m)(m-b) + B(a-b)(m-n) & \\ +A((n-a)(m-b) - (m-a)(n-b)) &\leq 0 \\ (U-T)(n-m)(m-b) + B(a-b)(m-n) + A(a-b)(n-m) &\leq 0 \\ (U-T)(n-m)(m-b) + (B-A)(a-b)(m-n) &\leq 0 \\ (n-m)\left((U-T)(m-b) + (B-A)(b-a)\right) &\leq 0 \\ (U-T)(m-b) + (B-A)(b-a) &\geq 0 \\ (B-A)(a-b) + (U-T)(b-m) &\leq 0 \end{aligned}$$

This is exactly (4.9).
So if (4.9) **holds**, we have $V \leq T + S \leq A + U$.
the min–plus convolution reads

$$(f * g)(t) = \begin{cases} m \cdot (t - (T+A)) & T+B \leq t \leq T+S \\ b \cdot (t - (T+B)) + a \cdot (B-A) & T+S < t \leq U+B \end{cases}$$

If (4.9) **is false**, we get instead for $A + U \leq T + S \leq V \leq U + B$.

$$(f * g)(t) = \begin{cases} m \cdot (t - (T+A)) & T+B \leq t \leq A+U \\ n \cdot (t - (A+U)) + m(U-T) & A+U < t \leq V \\ b \cdot (t - (T+B)) + a \cdot (B-A) & V < t \leq U+B \end{cases}$$

If (4.9) **is false**, we get for $A + U \leq T + S \leq U + B < V$.

$$(f * g)(t) = \begin{cases} m \cdot (t - (T+A)) & T+B \leq t \leq A+U \\ n \cdot (t - (A+U)) + m(U-T) & A+U < t \leq U+B \end{cases}$$

Note that we can not estimate in general, if $V \leq U + B$.
If we choose $b = 1, m = 4, a = 5$ and $(U - T) = k(B - A)$, $k > 1$, then we have

$$(a-n)(B-A) + (b-m)(U-T) = (B-A)(5-n-3k)$$

Choosing for example $k = 1.1$ and $n = 1.1$, this is positive, whereas e.g. the choice $k = 2$ and $n = 1.5$ yields a negative number.

$U + B < t$ We have

$$(f * g)(t) = \min\{f(t-A), b \cdot (t-(T+B)) + a \cdot (B-A), \Gamma_1, \eta_1, \eta_2\}$$

with

$$\begin{aligned} f(t-A) &= n \cdot (t-(A+U)) + m \cdot (U-T) \\ \Gamma_1 &= n \cdot (t-(A+U)) + m(U-T) \\ \eta_1 &= b \cdot (t-(B+U)) + m(U-T) + a(B-A) \\ \eta_2 &= b \cdot (t-(B+T)) + a(B-A) \end{aligned}$$

We can rewrite η_2

$$b \cdot (t-(B+T)) + a(B-A) = b \cdot (t-(B+U)) + b(U-T) + a(B-A)$$

Since $b < m$ and $U > T$ we immediately realize $\eta_2 \leq \eta_1$. So we have to decide for which t we have $\Gamma_1(t) \leq \eta_2(t)$.

$$n \cdot (t-(A+U)) + m(U-T) \leq b \cdot (t-(B+T)) + a(B-A)$$

We have already done this for $A+U \leq t \leq U+B$; we have $\Gamma_1(t) \leq \eta_2(t)$ iff $t \leq V$. If $V \leq U+B$, then we have

$$(f * g)(t) = b \cdot (t-(B+T)) + a(B-A) \qquad U+B < t$$

else we get

$$(f * g)(t) = \begin{cases} n \cdot (t-(A+U)) + m(U-T) & U+B < t \leq V \\ b \cdot (t-(B+T)) + a(B-A) & V < t \end{cases}$$

So we collect the results:
We set

$$\begin{aligned} S &:= \frac{B(a-b) + A(m-a)}{m-b} \\ V &:= \frac{T(m-b) + U(n-m) + B(a-b) + A(n-a)}{n-b} \end{aligned}$$

$$a \geq m \geq n > b \qquad B - A < U - T$$

$$(a-b)(B-A) + (b-m)(U-T) \leq 0$$

$$(f * g)(t) = \begin{cases} 0 & t \leq T+A \\ m \cdot (t-(T+A)) & T+A \leq t \leq T+S \\ b \cdot (t-(T+B)) + a \cdot (B-A) & T+S < t \end{cases}$$

$$a \geq m \geq n > b \qquad B - A < U - T$$

$$(a-b)(B-A) + (b-m)(U-T) > 0$$

$$(f * g)(t) = \begin{cases} 0 & t \leq T+A \\ m \cdot (t-(T+A)) & T+A \leq t \leq A+U \\ n \cdot (t-(A+U)) + m(U-T) & A+U < t \leq V \\ b \cdot (t-(T+B)) + a \cdot (B-A) & V < t \end{cases}$$

4.5 Realizing the pattern

In this section, we describe the pattern found after carefully looking at the results from the previous calculations.
We want to compute the min–plus–convolution of the following two functions f and g, where $A_i > A_j$ and $B_i > B_j$ if $i > j$:

$$f(t) = \begin{cases} 0 & t \leq A_0 \\ a_0 \cdot (t - A_0) & A_0 \leq t \leq A_1 \\ a_0 \cdot (A_1 - A_0) + a_1 \cdot (t - A_1) & A_0 \leq t \leq A_1 \\ \vdots & \\ a_0 \cdot (A_1 - A_0) + a_1 \cdot (A_2 - A_1) + \ldots + a_{n-1} \cdot (t - A_{n-1}) & A_{n-1} \leq t \leq A_n \\ a_0 \cdot (A_1 - A_0) + a_1 \cdot (A_2 - A_1) + \ldots + a_n \cdot (t - A_n) & A_n \leq t \end{cases}$$

$$g(t) = \begin{cases} 0 & t \leq B_0 \\ b_0 \cdot (t - B_0) & B_0 \leq t \leq B_1 \\ b_0 \cdot (B_1 - B_0) + b_1 \cdot (t - B_1) & B_0 \leq t \leq B_1 \\ \vdots & \\ b_0 \cdot (B_1 - B_0) + b_1 \cdot (B_2 - B_1) + \ldots + b_{n-1} \cdot (t - B_{n-1}) & B_{n-1} \leq t \leq B_n \\ b_0 \cdot (B_1 - B_0) + b_1 \cdot (B_2 - B_1) + \ldots + b_n \cdot (t - B_n) & B_n \leq t \end{cases}$$

We assume that the functions f and g are concave from A_0 respectively B_0 onwards, i.e. we have $a_i < a_j$ and $b_i < b_j$ for $j < i$. In other words, we assume $a_n < a_{n-1} < \ldots < a_1$ and $b_n < b_{n-1} < \ldots < b_1$. Furthermore we assume $A_0 \leq B_0$ without loss of generality.

Proposition 4.5 The min–plus–convolution of f and g is given by

$$(f * g)(t) = \inf_{0 \leq s \leq t} \{f(t-s) + g(s)\} = \min\{f(t - B_0), g(t - A_0)\}$$

Proof $(f * g)(t) = 0$ for $t \leq A_0 + B_0$ (e.g. Kirchner or other papers...).
Let $t > A_0 + B_0$ and assume that

$$f(t - B_0) \geq g(t - A_0)$$

Let x be such that $A_0 + B_0 + x < t$. Then we have the following inequalities:

$$\frac{f(t - B_0) - f(A_0)}{t - B_0 - A_0} \leq \frac{f(A_0 + x) - f(A_0)}{x} \tag{4.10}$$

$$\frac{g(t - A_0) - g(t - A_0 - x)}{x} \leq \frac{g(t - A_0) - g(B_0)}{t - B_0 - A_0} \tag{4.11}$$

$$\frac{f(t - B_0) - f(A_0)}{t - B_0 - A_0} \geq \frac{g(t - A_0) - g(B_0)}{t - B_0 - A_0} \tag{4.12}$$

Inequality (3) simply restates our assumption $f(t - B_0) \geq g(t - A_0)$.
Inequality (1) and (2) hold, since f respectively g are concave. We will give a proof

below.

We can write down a series of inequalities:

$$\begin{aligned}\frac{g(t-A_0)-g(t-A_0-x)}{x} &\leq \frac{g(t-A_0)-g(B_0)}{t-B_0-A_0}\\ &\leq \frac{f(t-B_0)-f(A_0)}{t-B_0-A_0}\\ &\leq \frac{f(A_0+x)-f(A_0)}{x}\end{aligned}$$

Mulitplying by $x > 0$ and rearranging we obtain

$$g(t-A_0)+f(A_0) \leq f(A_0+x)+g(t-A_0-x)$$

Setting $y := A_0 + x$ and recalling $f(A_0) = 0$ we have

$$(\dagger) \qquad g(t-A_0) \leq f(y)+g(t-y)$$

Proof of inequality (1)

We have $t > A_0 + B_0$. For $x \geq 0$ with $t \geq A_0 + B_0 + x$ we have $A_0 \leq A_0 + x \leq t - B_0$, hence $A_0 + x$ can be written as unique convex combination of the points A_0 and $t - B_0$. Therefore we need to solve the following:

$$A_0 + x = \alpha \cdot A_0 + (1-\alpha)\cdot(t-B_0)$$

A simple computation yields

$$\alpha = \frac{A_0+x-t+B_0}{A_0-t+B_0} = 1 - \frac{x}{t-A_0-B_0} < 1$$

Hence we have

$$1-\alpha = \frac{x}{t-A_0-B_0}$$

Now we use that f is concave. Recall, that a function η is concave on $[a,b]$, if for every $\alpha \in [0,1]$ the following inequality holds:

$$\alpha \cdot \eta(a) + (1-\alpha)\cdot\eta(b) \leq \eta(\alpha\cdot a + (1-\alpha)\cdot b)$$

Setting $a = A_0$ and $b = t - B_0$, we obtain

$$\begin{aligned}f(A_0+x) &\geq \alpha f(A_0) + (1-\alpha) f(t-B_0)\\ &= f(A_0) - \frac{x}{t-A_0-B_0}\cdot f(A_0) + \frac{x}{t-A_0-B_0} f(t-B_0) \qquad (4.13)\\ &= f(A_0) + \frac{x}{t-A_0-B_0}\cdot(f(t-B_0)-f(A_0)) \qquad (4.14)\end{aligned}$$

Rearraging yields inequality (1).

Proof of inequality (2)

We have $t > A_0 + B_0$. For $x \geq 0$ with $t \geq A_0 + B_0 + x$ we have $B_0 \leq t - A_0 - x \leq t - A_0$, hence $t - A_0 - x$ can be written as unique convex combination of the points B_0 and $t - A_0$. Therefore we need to solve the following:

$$t - A_0 - x = \alpha \cdot B_0 + (1 - \alpha) \cdot (t - A_0)$$

A simple computation yields

$$\alpha = \frac{-x}{B_0 + A_0 - t} = \frac{x}{t - A_0 - B_0} \leq 1$$

Now we use that g is concave. Recall, that a function η is concave on $[a, b]$, if for every $\alpha \in [0, 1]$ the following inequality holds:

$$\alpha \cdot \eta(a) + (1 - \alpha) \cdot \eta(b) \leq \eta(\alpha \cdot a + (1 - \alpha) \cdot b)$$

Setting $a = B_0$ and $b = t - A_0$, we obtain

$$\begin{aligned} g(t - A_0 - x) &\geq \alpha g(B_0) + (1 - \alpha) g(t - A_0) \\ &= \frac{x}{t - A_0 - B_0} \cdot g(B_0) + \left(1 - \frac{x}{t - A_0 - B_0}\right) \cdot g(t - A_0) \quad (4.15) \\ &= g(t - A_0) + \frac{x}{t - A_0 - B_0} \cdot (g(B_0) - g(t - A_0)) \quad (4.16) \end{aligned}$$

So we have

$$\frac{g(t - A_0 - x) - g(t - A_0)}{x} \geq \frac{g(B_0) - g(t - A_0)}{t - A_0 - B_0}$$

Multiplying by -1 yields inequality (2).

5 Conclusion

In this paper we considered possibilities to make the computation of the min–plus convolution easier. Our first ideas were to try to apply known results and techniques from convex analysis. The Fenchel–Transformation seemed to be a very promising tool. The computation of the Fenchel–Transformation is relatively easy and has a certain geometrical interpretation. By applying the Fenchel–Transformation the computation of the min–plus convolution is reduced to a pointwise addition. Also, we immediately have the inverse transformation; we simply have to apply the Fenchel–Transformation again. But in the course of our studies we found that it is not straightforward to extend the results we obtained for convex functions to nonconvex ones.

Our first idea, decomposing a nonconvex function into its convex parts and apply the Fenchel–Transformation on the convex parts did not work, since the Fenchel–Transformation was not compatible with the decomposition operation. Furthermore we showed that the Fenchel–Transformation as a mapping from the min–plus algebra to the min–plus algebra is nonlinear; so we started to look for a linear transformation. We were not able to devise one, we could only develop ideas and desireable properties of such a transformation.

In the last chapter we computed the min–plus convolution explicitly for certain rather easy nonconvex functions. We aimed at understanding the influence of the parameters of the functions on the convolution. We found a pattern which lead to a general equation to calculate the min–plus convolution of such functions.

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