ON NON-LINEAR SPACE-TIME BLOCK CODES

S. Sandhu and A. Paulraj

Stanford University
Stanford, CA 94305-9510
sumeet { apaulraj }@stanford.edu

K. Pandit

Technical University of Darmstadt
Darmstadt, Germany
pandit@hrzpub.tu-darmstadt.de

ABSTRACT

A number of space-time block codes have been proposed for the quasi-static, flat-fading channel with coherent receiver. All of these block codes are linear codes, i.e., the encoded codeword is a linear function of the input scalar symbols. Here we propose new *non-linear* space-time block codes, i.e., the codewords are non-linear functions of the input scalar symbols. We demonstrate a non-linear code that outperforms the corresponding linear code by 0.6 to 1.2 dB. We draw parallels between optimal non-linear code design and the well-known *simplex conjecture* for multi-dimensional AWGN codes. Finally we show that for certain non-uniformly distributed input symbols, the optimal signal set cannot be designed with linear modulation and non-linear modulation is essential for optimality. This has applications in the design of space-time trellis codes and concatenated space-time coding schemes.

1. INTRODUCTION

A number of space-time block codes have been proposed for the quasi-static, flat-fading channel with coherent receiver [1, 2, 3, 4, 5, 6]. All of these block codes are linear codes, i.e., the encoded codeword is a linear function of the input scalar symbols. Here we propose new *non-linear* space-time block codes, i.e., the codewords are non-linear functions of the input scalar symbols. We demonstrate a non-linear code that outperforms the corresponding linear code by 0.6 to 1.2 dB. We draw parallels between optimal non-linear code design and the well-known *simplex conjecture* for multi-dimensional AWGN codes. Finally we show that for certain non-uniformly distributed input symbols, the optimal signal set cannot be designed with linear modulation and non-linear modulation is essential for optimality. This has applications in the design of space-time trellis codes and concatenated space-time coding schemes.

2. CHANNEL AND DATA MODEL

Consider a system with M_r receive antennas and M_t transmit antennas. The channel is flat-fading and quasi-static. It is unknown at the transmitter and fully known at the receiver. The channel output corresponding to an input block spanning T time samples is

$$\mathbf{Y} = \mathbf{HX} + \mathbf{V} \tag{1}$$

where the received signal is $\mathbf{Y}$ ($M_r \times T$), the fading channel is $\mathbf{H}$ ($M_r \times M_t$), the encoded codeword is $\mathbf{X}$ ($M_t \times T$), and receiver noise is $\mathbf{V}$ ($M_r \times T$). When $\mathbf{H}$ is assumed to be i.i.d. Gaussian in the sequel, its entries are i.i.d. circular complex Gaussian random variables with variance 0.5 in each dimension, i.e. $\mathbf{H}_{ij} \sim \aleph_c(0,1)$. The entries of $\mathbf{V}$ are i.i.d. with $\mathbf{V}_{ij} \sim \aleph_c(0, N_0)$. The total average power transmitted on M_t antennas is E_s per sample time. Define $S = \frac{E_s}{N_0}$.

We will consider design of *non-linear* space-time block codes described as follows

$$\mathbf{X} = \sum_{k=1}^{K} \mathbf{A}_k x_k \tag{2}$$

where $\{x_k\}_{k=1}^{K}$ is a set of K real symbols that are non-linear functions of $K_i/2$ complex input symbols from a QAM constellation. The complex *modulation matrices* $\mathbf{A}_k$ are normalized as $\sum_{k=1}^{K} \|\mathbf{A}_k\|_F^2 = T$ to obtain total average transmit power of E_s per unit time. The complex input symbols are assumed to be uncoded and the encoder operates over them in a blockwise memoryless fashion. The receiver performs ML decoding on each $M_r \times T$ output block $\mathbf{Y}$.

For K_i input scalars $s_1, \ldots, s_{K_i}$ that correspond to the real and imaginary parts of $K_i/2$ complex input symbols, let $x_k = f_k(s_1, \ldots, s_{K_i})$ be one of K mappings to output scalars that are encoded using K modulation matrices. One way to perform non-linear mappings is to take all possible products of unique groups of input symbols [7] as follows

$$\begin{aligned} & \left[\; x_1, \; x_2, \; \ldots, \; x_{K_i}, \; x_{K_i+1}, \; \ldots, \; x_K \;\right] \\ = & \left[\; s_1, \; s_2, \; \ldots, \; s_{K_i}, \; s_1 s_2, \; \ldots, \; \prod_{k=1}^{K_i} s_k \;\right] \end{aligned}$$

where $K = 2^{K_i} - 1$ is the total number of products. The linear codes considered previously [8] consist of the first K_i

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elements on both sides of the equation, i.e., the identity map $x_k = s_k$ with $K = K_i$.

Note that when the input symbols s_k belong to a unit-norm BPSK constellation $(+1, -1)$, all products and therefore the nonlinear output symbols x_k also belong to BPSK. In addition, each nonlinear symbol possesses a uniform distribution over the BPSK constellation if the input is uniformly distributed. This can be extended to an M-PAM input constellation by carefully mapping the products back to elements in the M-PAM such that each x_k is distributed uniformly over the M-PAM [9]. The number of positive x_k is therefore equal to the number of negative x_k for $1 \leq k \leq K$, and Theorems 1 and 2 in [8] hold for the nonlinear case as well. That is, the optimal modulation matrices are still the pairwise skew-Hermitian matrices as follows

$$\mathbf{A}_k\mathbf{A}_l^* + \mathbf{A}_l\mathbf{A}_k^* = \mathbf{0} \qquad \text{for } 1 \leq k \neq l \leq K \quad (3)$$

3. LINEAR VERSUS NON-LINEAR CODES

In this section we will show that non-linear codes can improve performance over linear codes. Consider two codes based on the Alamouti matrices [3, 4], the first of which uses two matrices to encode 2 BPSK symbols and the second uses three matrices to encode 2 BPSK symbols and their product. The input symbols and error sequences for the first code, i.e., the linear code, are shown in Table 1. All are equally likely with probability $p_i = 1/4$. The input

p_i	$x^{(i)}$ / $x^{(j)}$	(-1 -1)	(-1 1)	(1 -1)	(1 1)
$\frac{1}{4}$	(-1 -1)	(0 0)	(0 -2)	(-2 0)	(-2 -2)
$\frac{1}{4}$	(-1 1)	(0 2)	(0 0)	(-2 2)	(-2 0)
$\frac{1}{4}$	(1 -1)	(2 0)	(2 -2)	(0 0)	(0 -2)
$\frac{1}{4}$	(1 1)	(2 2)	(2 0)	(0 2)	(0 0)

Table 1. Error sequences for input BPSK, 2 modulation matrices

symbols and error sequences for the second code, which is non-linear, are listed in Table 2.

$x^{(i)}$ / $x^{(j)}$	(-1 -1 1)	(-1 1 -1)	(1 -1 -1)	(1 1 1)
(-1 -1 1)	(0 0 0)	(0 -2 2)	(-2 0 2)	(-2 -2 0)
(-1 1 -1)	(0 2 -2)	(0 0 0)	(-2 2 0)	(-2 0 -2)
(1 -1 -1)	(2 0 -2)	(2 -2 0)	(0 0 0)	(0 -2 -2)
(1 1 1)	(2 2 0)	(2 0 2)	(0 2 2)	(0 0 0)

Table 2. Error sequences for input BPSK symbols and product, 3 modulation matrices

After appropriate scaling of the Alamouti matrices to normalize transmit power, the channel-averaged union bound [10] for the linear code simplifies as

$$P_{lin} = 2\left(1+\frac{S}{2}\right)^{-2M_r} + (1+S)^{-2M_r}$$

and that for the non-linear code simplifies to

$$P_{nonlin} = 3\left(1+\frac{2S}{3}\right)^{-2M_r}$$

Using Jensen's inequality on the convex function $(1+S)^{-2M_r}$ for $S > 0$, it can be shown that the non-linear code is better than the linear code as follows

$$\begin{aligned} P_{lin} &= 3\left(\frac{2}{3}\left(1+\frac{S}{2}\right)^{-2M_r} + \frac{1}{3}(1+S)^{-2M_r}\right) \\ &\geq 3\left(1+\frac{2S}{3}\right)^{-2M_r} = P_{nonlin} \qquad (4) \end{aligned}$$

The performance difference is a function of the SNR and number of receive antennas. It ranges from 0.6 dB to 1.2 dB for 1 to 3 receive antennas, respectively, and is plotted in Figure 1.

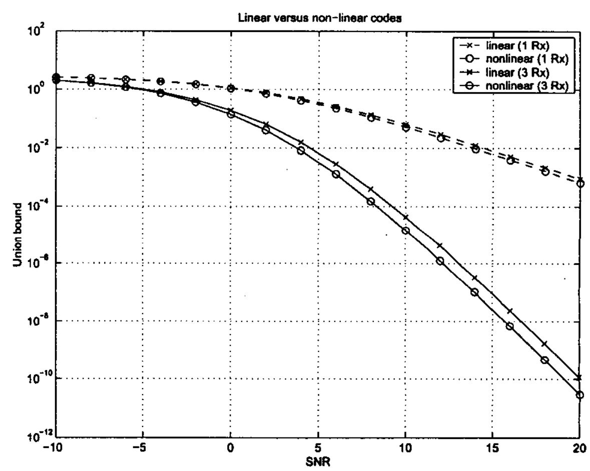

Fig. 1. Non-linear versus linear code

3.1. Relation to the simplex conjecture

The interesting change in going from Table 1 to 2 is that the four points that were vertices of a square in two dimensions move to the vertices of a tetrahedron in three dimensions. Addition of the third modulation matrix effectively adds a new signal dimension that helps make the points equidistant. Equidistant points are the vertices of a regular simplex, which is conjectured to be the optimal signal set for AWGN channels. In fact the regular simplex is known to minimize the union bound at all SNRs [11].

What is apparent here is that when optimal modulation matrices that satisfy (3) are used to map input symbols to space-time codewords, the problem of code design for the fading channel reduces to the problem of code design for the AWGN channel. Since optimal modulation matrices are very limited in existence, the question of how quasi-orthogonal modulation matrices affect code design remains to be answered.

4. NON-UNIFORM INPUT DISTRIBUTIONS

In this section we will show that non-linear codes may be necessary for optimal performance when the input symbols are not uniformly distributed. If the input scalar stream s_k is encoded using block or trellis codes, a block of $K_i/2$ such symbols may not consist of i.i.d. symbols and certain values of the block can be more likely than others. In such cases linear codes may not be optimal as we will show by an example.

Consider the simple case of (1×1) modulation matrices that are all normalized scalars. The two optimal modulation scalars are $1/\sqrt{2}$ and $j/\sqrt{2}$ as per (3). The resulting codewords are illustrated in Figure 2 and have a d_{min} of $\sqrt{2}$.

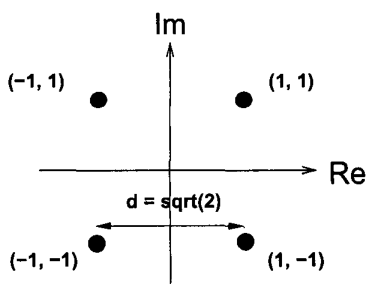

Fig. 2. Codewords for uniform input symbols

Consider the following variation on Table 1 obtained by setting the probability of input sequence $x^{(1)} = (-1, -1)$ equal to zero. The sequences $(-1, 1)$, $(1, -1)$ and $(1, 1)$ are equally likely with probability $\frac{1}{3}$.

p_i	$x^{(i)}$ / $x^{(j)}$	(-1 1)	(1 -1)	(1 1)
$\frac{1}{3}$	(-1 1)	(0 0)	(-2 2)	(-2 0)
$\frac{1}{3}$	(1 -1)	(2 -2)	(0 0)	(0 -2)
$\frac{1}{3}$	(1 1)	(2 0)	(0 2)	(0 0)

Table 3. Non-uniform input sequences for BPSK and K = 2

In this case, if the optimal modulation scalars for uniform symbols are used, we obtain the constellation in Figure 3, which has the same d_{min} as the previous constellation.

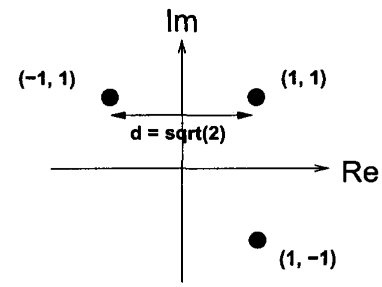

Fig. 3. Standard codewords for non-uniform input symbols

This is not the best possible constellation, however, and the d_{min} can be further increased to $\sqrt{3}$ by placing the symbols in a simplex as shown in Figure 4. What is interesting

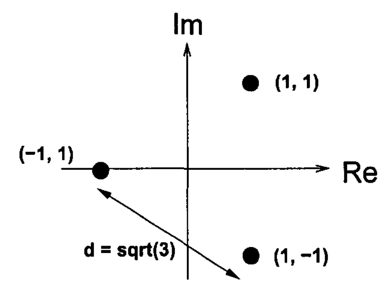

Fig. 4. Optimal codewords for non-uniform input symbols

in this case is that the only way to map these three symbols to the simplex is via a non-linear mapping as shown below

$$
\begin{array}{ccc} \mathbf{x} & = & \mathbf{S}\mathbf{a} \end{array}
$$

$$
\begin{bmatrix} e^{j\frac{\pi}{3}} \\ -1 \\ e^{-j\frac{\pi}{3}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2}e^{j\frac{2\pi}{3}} \\ \frac{1}{2}e^{j\frac{\pi}{3}} \end{bmatrix} \quad (5)
$$

where $\mathbf{S}$ is the matrix of all possible inputs, $\mathbf{a}$ is the vector of modulation scalars, and $\mathbf{x}$ is the output. The first two columns of $\mathbf{S}$ contain the values of two input scalars and the third column consists of products of two input scalars, which is a nonlinear mapping of the input symbols. An easy way to prove that this mapping cannot be achieved via linear modulation is to observe that two points are multiples of each other, i.e., $(-1, 1) = -(1, -1)$, but none of the points in the simplex are such multiples.

Non-uniform input symbols are likely to arise in concatenated systems where the input stream has been encoded by an outer code. The practical applications of non-linear codes for realistic code statistics remain to be seen.

5. ACKNOWLEDGMENT

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